Probability Maximization via Minkowski Functionals: Convex Representations and Tractable Resolution

Abstract

In this paper, we consider the maximization of a probability $\mathbb{P}\{\zeta \mid \zeta \in \mathbf{K}(\mathbf{x})\}$ over a closed and convex set $\mathcal{X}$, a special case of the chance-constrained optimization problem. We define $\mathbf{K}(\mathbf{x})$ as $\mathbf{K}(\mathbf{x}) \triangleq \{\zeta \in \mathcal{K} \mid c(\mathbf{x},\zeta) \geq 0\}$ where ζ is uniformly distributed on a convex and compact set $\mathcal{K}$ and $c(\mathbf{x},\zeta)$ is defined as either $c(\mathbf{x},\zeta) \triangleq 1-|\zeta^T\mathbf{x}|^m$, $m \geq 0$ (Setting A) or $c(\mathbf{x},\zeta) \triangleq T\mathbf{x}-\zeta$ (Setting B). We show that in either setting, by leveraging recent findings in the context of non-Gaussian integrals of positively homogeneous functions, $\mathbb{P}\{\zeta \mid \zeta \in \mathbf{K}(\mathbf{x})\}$ can be expressed as the expectation of a suitably defined function $F(\mathbf{x},\xi)$ with respect to an appropriately defined Gaussian density (or its variant), i.e. $\mathbb{E}_{\tilde{p}}[F(\mathbf{x},\xi)]$. Aided by a recent observation in convex analysis, we then develop a convex representation of the original problem requiring the minimization of $g(\mathbb{E}[F(\mathbf{x},\xi)])$ over $\mathcal{X}$ where g is an appropriately defined smooth convex function. Traditional stochastic approximation schemes cannot contend with the minimization of $g(\mathbb{E}[F(\bullet,\xi)])$ over $\mathcal{X}$, since conditionally unbiased sampled gradients are unavailable. We then develop a regularized variance-reduced stochastic approximation (**r-VRSA**) scheme that obviates the need for such unbiasedness by combining iterative regularization with variance-reduction. Notably, (**r-VRSA**) is characterized by both almost-sure convergence guarantees, a convergence rate of $\mathcal{O}(1/k^{1/2-a})$ in expected sub-optimality where $a > 0$, and a sample complexity of $\mathcal{O}(1/\epsilon^{6+\delta})$ where $\delta > 0$. This is joint work with Ibrahim Bardakci, Afrooz Jalilzadeh, and Constantino Lagoa.