Sandia National Laboratories

Randomized Sketcing for Low-Memory Dynamic Optimization

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LDRD

ENERGY NNSA

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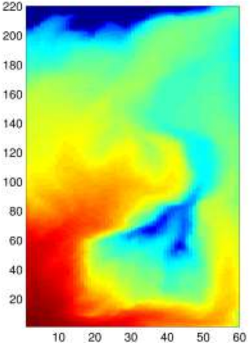
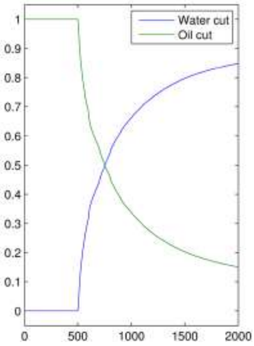
Outline

1. Motivating Applications
2. General Problem Formulation
3. Randomize Sketching for State Compression
4. Fixed Rank Analysis
5. Adaptive Rank Algorithm
6. Numerical Results

Operational Applications

Secondary Oil Recovery





$$\max_{v,p,s,q} \ \mathrm{NPV}(v,p,s,q)$$

subject to

$$v = -\mathbf{K}\lambda(s)(\nabla p - \rho G)$$

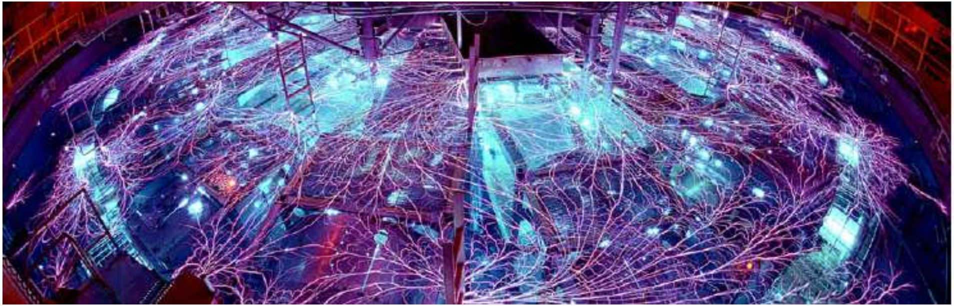
$$\nabla \cdot v = q/\rho$$

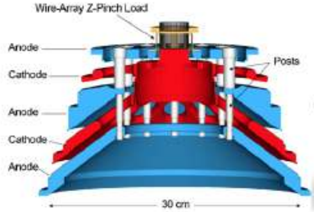
$$\phi\, \partial_t s + \nabla \cdot (f(s)v) = q$$

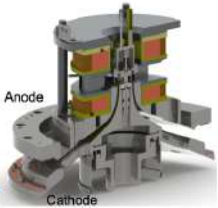
1. **Goal:** Determine injection rates that maximize the net present value of the reservior.
2. **Problem Size:** The reservoir can span 10+ km per side $\Longrightarrow$ 1000km^3
   A regular grid with 5m$\times$5m$\times$5m cells $\Longrightarrow$ 8 billion cells!
3. **Similar Applications:** Gas pipeline and energy network operations, financial portfolio optimization, optimal control of hypersonic vehicles, etc.

Optimal Design

Convolute Design for the Z-Machine (Inertial Confinement Fusion)





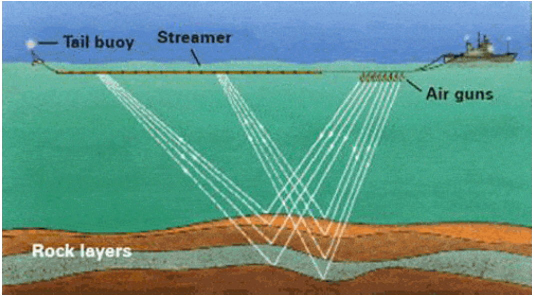


Goal: Minimize power loss – **current design experiences 20% power loss**

Similar Applications: Designing elastic structures with topology optimization, shape optimization for airplane wing design, etc.

Inverse Problems

Seismic Exploration



Courtesy of Schlumberger

$$\min_{u,\theta} \ \frac{1}{2}\|u-d\|^2 + \alpha\wp(\theta)$$

subject to

$$\partial_t\left(\varrho(\theta)\,\partial_t u\right) - \nabla\cdot\left(\mathbf{K}(\theta) : \varepsilon\right) = s$$

$$\varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^\top)$$

1. **Goal:** Determine subsurface rock properties that minimize data misfit.
2. **Problem Size:** **Typical marine surveys require many TBs of memory!**
3. **Similar Applications:** Parameter calibration, data assimilation, supervised learning for neural networks, etc.

Discrete Dynamic Optimization

Full-Space Formulation

$$
\begin{aligned}
\min_{z_n\in\mathbb{R}^m,\ u_n\in\mathbb{R}^M} \quad & \sum_{n=1}^{N} f_n(u_{n-1}, u_n, z_n) \\
\text{subject to} \quad & c_n(u_{n-1}, u_n, z_n) = 0, \quad n = 1, \ldots, N
\end{aligned}
$$

where $f_n : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^m \to \mathbb{R}$ is the ***cost function at time*** t_n,

$c_n : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^m \to \mathbb{R}^M$ is the ***dynamic constraint at time*** t_n,

$u_n \in \mathbb{R}^M$ is the ***state/trajectory at time*** t_n, and

$z_n \in \mathbb{R}^m$ is the ***control/design/parameters at time*** t_n.

E.g., continuous dynamic problem discretized in time (and space).

Our methods also work for time-independent control/design/parameters.

Discrete Dynamic Optimization

The **KKT conditions** for the full-space dynamic optimization problem are

$$c_n(u_{n-1}, u_n, z_n) = 0$$

$$(\mathsf{d}_2 c_N(u_{N-1}, u_N, z_N))^\top \lambda_N = -\,\mathsf{d}_2 f_N(u_{N-1}, u_N, z_N)$$

$$(\mathsf{d}_2 c_n(u_{n-1}, u_n, z_n))^\top \lambda_n + (\mathsf{d}_1 c_{n+1}(u_n, u_{n+1}, z_{n+1}))^\top \lambda_{n+1} = -\,\mathsf{d}_2 f_N(u_{n-1}, u_n, z_n) - \mathsf{d}_1 f_{n+1}(u_n, u_{n+1}, z_{n+1})$$

$$\mathsf{d}_3 f_n(u_{n-1}, u_n, z_n) + (\mathsf{d}_3 c_n(u_{n-1}, u_n, z_n))^\top \lambda_n = 0, \quad n = 1, \ldots, N.$$

The new variables $\lambda_1, \ldots, \lambda_N$ are **Lagrange multipliers** for the dynamic constraint.

Most algorithms (e.g., SQP, AL, etc.) require storage of $\{u_n\}_{n=1}^N$, $\{\lambda_n\}_{n=1}^N$ and $\{z_n\}_{n=1}^N$.

Storage Requirement: $\mathcal{O}(N(2M+m))$ floating point numbers

$\Longrightarrow$ Can be **huge** (e.g., TBs of memory)!

Discrete Dynamic Optimization

If $c_n(u_{n-1}, u_n, z_n) = 0$ has a unique solution $u_n = S_n(u_{n-1}, z_n)$ for each $z \in Z_{\mathrm{ad}}$ and fixed initial state $u_0 \in \mathbb{R}^M$, $n = 1, \ldots, N$, then we can equivalently solve

$$\boxed{\min_{Z=\{z_n\}_{n=1}^N \in \mathbb{R}^{mN}} \quad F(Z)}$$

where the ***reduced*** objective function F is given by

$$\boxed{F(Z) := f_1(u_0, S_1(u_0, z_1), z_1) + \sum_{n=2}^{N} f_n(S_{n-1}(u_{n-2}, z_{n-1}), S_n(u_{n-1}, z_n), z_n).}$$

Storage Requirement: $\mathcal{O}(Nm)$ floating point numbers — typically, $m \ll M$.

Discrete Dynamic Optimization

Gradient Computation:

$$(\nabla F(Z))_n = \mathsf{d}_3 f_n(u_{n-1}, u_n, z_n) + (\mathsf{d}_3 c_n(u_{n-1}, u_n, z_n))^\top \lambda_n$$

where $\lambda_n \in \mathbb{R}^M$, $n = 1, \ldots, N$ solves the ***backward-in-time*** **adjoint equaiton**

$$\begin{aligned}
(\mathsf{d}_2 c_N(u_{N-1}, u_N, z_N))^\top \lambda_N &= -\,\mathsf{d}_2 f_N(u_{N-1}, u_N, z_N) \\
(\mathsf{d}_2 c_n(u_{n-1}, u_n, z_n))^\top \lambda_n &= -\,\mathsf{d}_2 f_N(u_{n-1}, u_n, z_n) - \mathsf{d}_1 f_{n+1}(u_n, u_{n+1}, z_{n+1}) \\
&\quad - (\mathsf{d}_1 c_{n+1}(u_n, u_{n+1}, z_{n+1}))^\top \lambda_{n+1}
\end{aligned}$$

Gradient computations requires entire state trajectory to solve adjoint equation!

Discrete Dynamic Optimization

1. **Checkpointing:** Store a handful of state snapshots in memory or to hard disk. Recompute state from snapshots during adjoint computation.

   Griewank and Walther, *Algorithm 799: Revolve: An implementation of checkpointing for scientific computing and optimal control*, 2000.

   Significant increase in computational cost!

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2. **Model Reduction:** Project dynamic equation onto a low-order basis of the solution space (e.g., balanced truncation, POD, reduced basis, etc.).

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3. **State Compression:** Compress dynamic trajectory using numerical compression techniques including nested meshes, PCA, Gramm-Schmidt and DFT.

   Cyr, Shadid and Wildey, *Towards efficient backward-in-time adjoint computations using data compression techniques*, 2015.

   Can be physics-dependent and often compresses time/space independently!

Randomized Sketching

We apply **randomized sketching** to compress the $M \times N$ state trajectory matrix

$$U = (u_1 \mid \cdots \mid u_N) \in \mathbb{R}^{M \times N}.$$

Goal: Generate an ***accurate*** rank-r approximation of U using $\mathcal{O}(r(M+N))$ storage.

Matrix Sketching: Given the four **random linear dimension reduction maps**

$$\Upsilon \in \mathbb{R}^{k \times M}, \qquad \Omega \in \mathbb{R}^{k \times N}, \qquad \Phi \in \mathbb{R}^{s \times M}, \qquad \text{and} \qquad \Psi \in \mathbb{R}^{s \times N},$$

where $k := 2r + 1$ and $s = 2k + 1$, we define the sketch as the following three matrices

$$X := \Upsilon U \in \mathbb{R}^{k \times N} \qquad \text{the co-range sketch;}$$

$$Y := U \Omega^\top \in \mathbb{R}^{M \times k} \qquad \text{the range sketch;}$$

$$Z := \Phi U \Psi^\top \in \mathbb{R}^{s \times s} \qquad \text{the core sketch.}$$

Tropp, Yurtserver, Udell and Cevher, *Streaming low-rank matrix approximation with an application to scientific simulation*, 2019.

Randomized Sketching

Choice of Random Matrices: **Standard normal**, scrambled subsampled randomized Fourier transform, sparse sign, . . .

Intuition:

- The range sketch $Y = U\Omega^\top$ consists of columns $U\omega_i$, which are independent random samples from $\mathrm{range}(U)$.

  Roughly speaking, Y captures the action of the ***top*** k ***left singular vectors***.

- Similarly, the co-range sketch $X = \Upsilon U$ consists of rows $\upsilon_i^\top U$, which are independent random samples from $\mathrm{range}(U^\top)$.

  Roughly speaking, X captures the action of the ***top*** k ***right singular vectors***.

- ***Roughly speaking***, the core sketch $Z = \Phi U \Psi^\top$ captures the ***top*** k ***singular values***.

Online State Sketching:

Require: $X = 0$, $Y = 0$, $Z = 0$

1: **for** n=1,...,N **do**
2: Given u_{n-1} and z_n, solve $c_n(u_{n-1}, u_n, z_n) = 0$ for u_n
3: Update $X \leftarrow X + \Upsilon u_n e_n^\top$
4: Update $Y \leftarrow Y + u_n(\Omega e_n)^\top$
5: Update $Z \leftarrow Z + (\Phi u_n)(\Psi e_n)^\top$
6: **end for**

Storage Requirement: Sketching requires $k(M+N) + s^2$ floating point numbers.

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Recovery:

1: $(Q, R_2) \leftarrow \mathtt{qr}(Y, 0)$ $\qquad Q \in \mathbb{R}^{M\times k},\quad R_2 \in \mathbb{R}^{k\times k}$
2: $(P, R_1) \leftarrow \mathtt{qr}(X^\top, 0)$ $\qquad P \in \mathbb{R}^{N\times k},\quad R_1 \in \mathbb{R}^{k\times k}$
3: $C \leftarrow (\Phi Q)^\dagger Z((\Psi P)^\dagger)^\top$ $\qquad C \in \mathbb{R}^{k\times k}$
4: $W \leftarrow CP^\top$ $\qquad W \in \mathbb{R}^{k\times N}$
5: $u_n \approx QWe_n$

Storage Requirement: Recovery requires $k(M+N)+k^2$ floating point numbers.

Theorem: Sketching Error Bound

Let $\{\{U\}\}_r$ denote the sketch of U associated with the rank parameter r. Then,

$$\mathbb{E}_{\Upsilon,\Omega,\Phi,\Psi}\|U - \{\{U\}\}_r\|_F \leq \sqrt{6}\Bigg(\sum_{i \geq r+1} \sigma_i^2(U)\Bigg)^{\frac{1}{2}}.$$

The error is *expected* to be slightly larger than the best rank-r approximation!

Similar results exist for the probability of large deviation.

Tropp, Yurtserver, Udell and Cevher, *Streaming low-rank matrix approximation with an application to scientific simulation*, 2019.

Solve Adjoint Equation and Compute Gradient:

Require: Sketched state $\{\{U\}\}_r$.

1: Reconstruct $\tilde{u}_{N-1}$ and $\tilde{u}_N$ from $\{\{U\}\}_r$.
2: Compute $\tilde{\lambda}_N$ that solves $(\mathrm{d}_2 c_N(\tilde{u}_{N-1}, \tilde{u}_N, z_N))^\top \tilde{\lambda}_N = -\mathrm{d}_2 f_N(\tilde{u}_{N-1}, \tilde{u}_N, z_N)$.
3: Set $g_N = \mathrm{d}_3 f_N(\tilde{u}_{N-1}, \tilde{u}_N, z_N) + (\mathrm{d}_3 c_N(\tilde{u}_{N-1}, \tilde{u}_N, z_N))^\top \tilde{\lambda}_N$.
4: **for** $n = N-1, \ldots, 1$ **do**
5: Reconstruct $\tilde{u}_{n-1}$ from $\{\{U\}\}_r$.
6: Compute $\tilde{\lambda}_n$ that solves

$$(\mathrm{d}_2 c_n(\tilde{u}_{n-1}, \tilde{u}_n, z_n))^\top \tilde{\lambda}_n = -\,\mathrm{d}_2 f_n(\tilde{u}_{n-1}, \tilde{u}_n, z_n) - \mathrm{d}_1 f_{n+1}(\tilde{u}_n, \tilde{u}_{n+1}, z_{n+1}) \\ - (\mathrm{d}_1 c_{n+1}(\tilde{u}_n, \tilde{u}_{n+1}, z_{n+1}))^\top \tilde{\lambda}_{n+1}.$$

7: Set $g_n = \mathrm{d}_3 f_n(\tilde{u}_{n-1}, \tilde{u}_n, z_n) + (\mathrm{d}_3 c_n(\tilde{u}_{n-1}, \tilde{u}_n, z_n))^\top \tilde{\lambda}_n$.
8: **end for**

Randomized Sketching

Define the combined objective function and constraint by

$$f(U,Z) = \sum_{i=1}^{N} f_i(u_{i-1}, u_i, z_i) \qquad \text{and} \qquad c(U,Z) = \begin{pmatrix} c_1(u_0, u_1, z_1) \\ \vdots \\ c_N(u_{N-1}, u_N, z_N) \end{pmatrix}.$$

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Assumptions:

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$$\sigma_0 \le \sigma_{\min}(\mathrm{d}_1 c(U,Z)) \le \sigma_{\max}(\mathrm{d}_1 c(U,Z)) \le \sigma_1;$$

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3. The functions $\mathrm{d}_1 c(U,Z)$, $\mathrm{d}_2 c(U,Z)$, $\mathrm{d}_1 f(U,Z)$ and $\mathrm{d}_2 f(U,Z)$ are Lipschitz continuous on $\mathcal{U} \times \mathcal{Z}$ with respect to U and the Lipschitz modulii are independent of $Z \in \mathcal{Z}$.

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Under these assumptions, we can show that

$$\|g - \nabla F(Z)\| \leq \kappa_1 \underbrace{\|c(\{\{U\}\}_r, Z)\|}_{\text{state residual}} \qquad \text{and} \qquad \mathbb{E}\|g - \nabla F(Z)\| \leq \sqrt{6}\kappa \Big(\sum_{i \geq r+1} \sigma_i^2(U) \Big)^{\frac{1}{2}}.$$

Fixed Rank Approach

- **Idea:** Use approximate gradient within a **derivative-based optimization algorithm**. For example, gradient descent, nonlinear CG, Newton–Krylov method, etc.
- If the rank parameter r is sufficiently large, then gradient error **may be negligible**.
- For Newton–Krylov, applying the Hessian to a vector requires **a linearized forward-in-time solve** and **a linearized backward-in-time solve**.
- Can **approximately** apply the Hessian to a vector by **sketching the adjoint as well as the trajectories from the additional solves**.

Fixed Rank Approach

Basic Trust-Region Method

Require: Initial guess $Z^{(0)} \in \mathbb{R}^{Nm}$ and target rank r. Algorithmic parameters: $\Delta^{(0)} > 0$ and $0 < \eta_0 \leq \eta_1 < \eta_2 < 1$.

1: **for** $\ell = 0, 1, 2, \ldots$ **do**

2: Solve state equation and sketch solution $\{\{U^{(\ell)}\}\}_r$.

3: Compute gradient approximation $g^{(\ell)} \approx \nabla F(Z^{(\ell)})$ using sketched state $\{\{U^{(\ell)}\}\}_r$.

4: Choose $B^{(\ell)} \in \mathbb{R}^{Nm \times Nm}$ to be an approximation of the Hessian $\nabla^2 F(Z^{(\ell)})$.

5: Compute $S^{(\ell)} \in \mathbb{R}^{Nm}$ that approximately solves the trust-region subproblem

$$\min_{S \in \mathbb{R}^{Nm}} \frac{1}{2}\langle S, B^{(\ell)} S\rangle + \langle S, g^{(\ell)}\rangle \quad \text{subject to} \quad \|S\| \leq \Delta^{(\ell)}.$$

6: Compute the ratio of actual and predicted reduction

$$\rho^{(\ell)} = \frac{F(Z^{(\ell)}) - F(Z^{(\ell)} + S^{(\ell)})}{-\frac{1}{2}\langle S^{(\ell)}, B^{(\ell)} S^{(\ell)}\rangle - \langle S^{(\ell)}, g^{(\ell)}\rangle}.$$

7: Set $Z^{(\ell+1)} = \begin{cases} Z^{(\ell)} & \text{if } \rho^{(\ell)} \leq \eta_0 \\ Z^{(\ell)} + S^{(\ell)} & \text{otherwise.} \end{cases}$

8: Choose $\Delta^{(\ell+1)} \in \begin{cases} (0, \Delta^{(\ell)}) & \text{if } \rho^{(\ell)} \leq \eta_1 \\ \{\Delta^{(\ell)}\} & \text{if } \rho^{(\ell)} \in (\eta_1, \eta_2) \\ (\Delta^{(\ell)}, \infty) & \text{otherwise.} \end{cases}$

9: **end for**

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Algorithm may not converge becuase gradient is inexact!

Adaptive Rank Algorithm

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   A priori choice of r is often difficult for practical applications!

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2. To ensure convergence of the TR method, gradient inexactness must be controlled

$$\|g^{(\ell)} - \nabla F(Z^{(\ell)})\| \le \kappa_0 \min\{\|g^{(\ell)}\|, \Delta^{(\ell)}\}.$$

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3. Satisfy gradient inexactness condition by monitoring state residual

$$\|g^{(\ell)} - \nabla F(Z^{(\ell)})\| \leq \kappa_1 \|c(\tilde{U}^{(\ell)}, Z^{(\ell)})\| \quad \Longrightarrow \quad \|c(\tilde{U}^{(\ell)}, Z^{(\ell)})\| \leq \kappa_0' \min\{\|g^{(\ell)}\|, \Delta^{(\ell)}\}.$$

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4. Evaluate state residual while solving adjoint equation.
   If bound is violated, increase r, re-solve state equation and re-sketch state!

Adaptive Rank Algorithm

1. Fixed-rank algorithm may not converge unless the rank r is chosen sufficiently large.
   A priori choice of r is often difficult for practical applications!

2. To ensure convergence of the TR method, gradient inexactness must be controlled

$$\|g^{(\ell)} - \nabla F(Z^{(\ell)})\| \leq \kappa_0 \min\{\|g^{(\ell)}\|, \Delta^{(\ell)}\}.$$

   Heinkenschloss and Vicente, *Analysis of incexact SQP algorithms*, 2001.

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4. Evaluate state residual while solving adjoint equation.
   If bound is violated, increase r, re-solve state equation and re-sketch state!

5. Hopefully sufficiently large r is determined after a handful of updates.
   May run out of memory if rank of state is too large!

Theorem: Convergence of Adaptive-Rank Trust-Region Method

Suppose there exists a bounded, open, convex set $\mathcal{Z} \subset \mathbb{R}^{Nm}$ such that $Z^{(\ell)} \in \mathcal{Z}$ for all ℓ. Suppose F is bounded below and twice continuously differentiable on $\mathcal{Z}$. Moreover, assume that $\nabla^2 F(\cdot)$ is uniformly bounded on $\mathcal{Z}$ and that $B^{(\ell)}$ is bounded independently of ℓ. Then,

$$\liminf_{\ell\to\infty} \|g^{(\ell)}\| = \liminf_{\ell\to\infty} \|\nabla F(Z^{(\ell)})\| = 0.$$

For detailed analysis of inexact trust-region methods see:

Carter, *On the global convergence of trust region algorithms using inexact gradient information*, 1991.

Conn, Gould and Toint, *Trust Region Methods*, 2000.

Heinkenschloss and Vicente, *Analysis of incexact SQP algorithms*, 2001.

Kouri and Ridzal, *Inexact trust-region methods for PDE-constrained optimization*, 2018.

Adaptive Rank Algorithm

"Simple" Control Constraints: Constraints with ***easy-to-compute*** projections such as

$$a_0 \le CZ \le a_1 \qquad \text{and} \qquad DZ = b.$$

Only need to modify trust-region subproblem to account for constraints in the set Z_{ad}, i.e.,

$$\min_{S \in \mathbb{R}^{Nm}} \ \frac{1}{2}\langle S, B^{(\ell)} S\rangle + \langle S, g^{(\ell)}\rangle \quad \text{subject to} \quad Z^{(\ell)} + S \in Z_{\mathrm{ad}}, \quad \|S\| \le \Delta^{(\ell)}.$$

Garreis and Ulbrich, *An inexact trust-region algorith for constrained problems in Hilbert space and its application to the adaptive solution of optimal control problems with PDEs*, 2019.

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Garreis and Ulbrich, *An inexact trust-region algorith for constrained problems in Hilbert space and its application to the adaptive solution of optimal control problems with PDEs*, 2019.

General Constraints: Use augmented Lagrangian, log barrier, etc. to penalize general nonlinear constraints such as

$$g(U,Z) = 0 \qquad \text{and} \qquad h(U,Z) \le 0.$$

Apply the adaptive rank algorithm to solve the penalized subproblems

$$\min_{Z \in \mathbb{R}^{Nm}} \ F(Z) + \wp^{(\ell)}(Z).$$

Example Application

Let $\mathcal{L}$ be a uniformly bounded and coercive linear operator, B a bounded linear operator and β a continuous linear functional. Consider the optimal control problem

$$\begin{aligned} \min_{u,\, z} \; & \frac{1}{2}\int_0^T \|u - w\|^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \|z\|^2 \,\mathrm{d}t \\ \text{subject to} \quad & \partial_t u + \mathcal{L}(t)u = \beta(t) + Bz \\ & u(0) = u_0. \end{aligned}$$

Discretization

1. Continuous finite elements in space and implicit Euler in time.
2. Similar results hold for trapezoidal (Crank-Nicolson) and explicit Euler.

Stability Estimates: C_n^1, C_n^2 are positive constants that depend on the time step δt_n

$$\|\tilde{u}_n - u_n\| \le C_n^1 \sum_{i=1}^n \|c_i(\tilde{u}_{i-1}, \tilde{u}_i, z_i)\| \quad \text{and} \quad \|\tilde{\lambda}_n - \lambda_n\| \le C_n^2 \sum_{i=1}^n \|c_i(\tilde{u}_{i-1}, \tilde{u}_i, z_i)\|.$$

Using these estimates, we can prove convergence of adaptive rank algorithm!

Numerical Results

Let $D = (0, 0.5) \times (0, 0.2)$ with boundary ∂D and consider the optimal control problem

$$\begin{aligned} &\min_{u,\,z} \frac{1}{2}\int_0^T \int_D |u-1|^2 \,\mathrm{d}x\mathrm{d}t + \frac{\alpha}{2}\int_0^T \int_D |z|^2 \,\mathrm{d}x\mathrm{d}t \\ &\text{subject to} \quad \partial_t u - \gamma \Delta u + \mathbf{v}\cdot\nabla u + u = \beta + z \qquad \text{in } (0,T)\times D \\ &\qquad\qquad\qquad\qquad \gamma \nabla u \cdot n = 0 \qquad \text{on } (0,T)\times \partial D \\ &\qquad\qquad\qquad\qquad u(0) = 0 \qquad \text{in } D, \end{aligned}$$

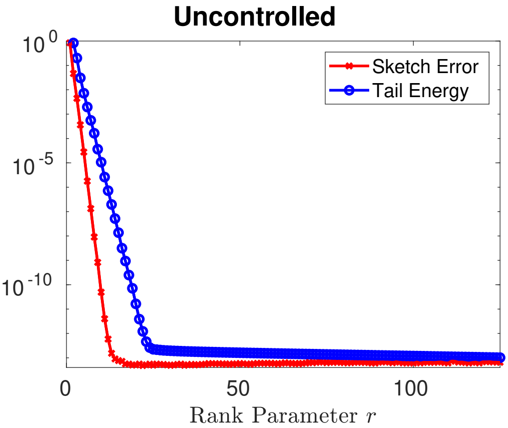
where $\alpha = 10^{-4}$, $\gamma = 0.1$, $\mathbf{v}(x) = (7.5 - 2.5x_1, 2.5x_2)^\top$, and

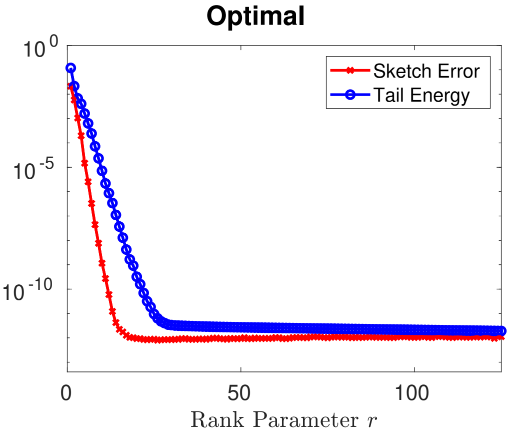
$$\beta(x) = \begin{cases} 1 & \text{if } \|x - (0.1, 0.1)^\top\| \le 0.07 \\ 0 & \text{otherwise.} \end{cases}$$

Spatial Discretization: Q1 finite elements on a uniform mesh of 60×20 quadrilaterals.

Temporal Discretization: Implicit Euler with 500 equal time steps.

Numerical Results





Sketching error averaged over 20 realizations and **tail energy** for the uncontrolled state (left) and the optimal state (right). Recall that the rank of the sketch is $k = 2r + 1$.

Numerical Results

Termination Criterion: $\|g^{(\ell)}\| \le 10^{-7}$ or $\ell > 20$.

Rank Increase Function: $r \leftarrow \max\{r+2, \lceil (b - \log\tau)/a \rceil\}$ with $a = 2.6125$, $b = 2.4841$.

rank	objective	iter	nstate	nadjoint	iterCG	compression
*1	5.544040e-4	20	21	11	196	118.79
2	5.528490e-4	5	6	6	151	70.96
3	5.528490e-4	4	5	5	78	50.46
4	5.528490e-4	4	5	5	67	38.08
5	5.528490e-4	4	5	5	59	31.83
Adaptive	5.528490e-4	4	8	5	65	23.14
Full	5.528490e-4	4	5	5	53	1.00

*The rank 1 experiment exceeded the maximum number of iterations.

Numerical Results

Linear Parabolic Control

	iter	value	gnorm	snorm	delta	iterCG	rank	rnorm
Adaptive	0	5.446e-2	5.990e-3	---	1.000e+1	---	1	2.707e-4
	1	1.375e-2	2.205e-3	1.000e+1	2.500e+1	1	1	1.990e-4
	2	1.475e-3	1.408e-4	2.499e+1	6.250e+1	5	5	6.700e-7
	3	5.531e-4	6.431e-7	4.077e+1	1.563e+1	27	7	3.893e-9
	4	5.528e-4	5.039e-9	1.059e+0	3.906e+2	32	7	1.508e-9
Full	0	5.446e-2	5.989e-3	---	1.000e+1	---	---	---
	1	1.375e-2	2.201e-3	1.000e+1	2.500e+1	1	---	---
	2	1.472e-3	1.401e-4	2.500e+1	6.250e+1	5	---	---
	3	5.538e-4	1.361e-6	4.051e+1	1.563e+1	19	---	---
	4	5.528e-4	8.416e-9	2.178e+0	3.906e+2	28	---	---

Adpative-rank algorithm reduced memory requirement 23.14-fold!

We consider the optimal flow control problem

$$\min_z \int_0^T \left\{ \int_{\partial C} \left(\frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}}{\partial n} - pn \right) \cdot (z\tau - \mathbf{v}_\infty)\,\mathrm{d}x + \frac{\alpha}{2} z(t)^2 \right\} \mathrm{d}t,$$

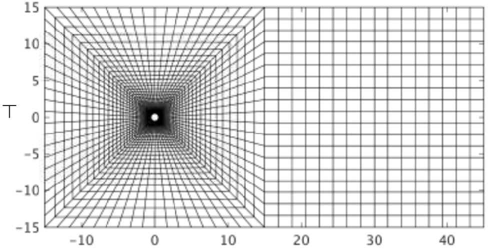
where the velocity/pressure $(\mathbf{v}, p) : [0,T] \times D \to \mathbb{R}^2 \times \mathbb{R}$ solves the Navier-Stokes equations

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} - \frac{1}{\mathrm{Re}} \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p &= 0 && \text{in } (0,T) \times D \setminus C \\
\nabla \cdot \mathbf{v} &= 0 && \text{in } (0,T) \times D \setminus C \\
\frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}}{\partial n} - pn &= 0 && \text{on } (0,T) \times \Gamma_{\mathrm{out}} \\
\mathbf{v} &= \mathbf{v}_\infty && \text{on } (0,T) \times \partial D \setminus \Gamma_{\mathrm{out}} \\
\mathbf{v} &= z\tau && \text{on } (0,T) \times \partial C
\end{aligned}$$

Goal: Minimize the **power required to overcome the drag** on C by rotating C.

$$\mathbf{v}_2 = 0$$

$\mathbf{v} = (0,1)^\top$



Γ_{out}

$$\mathbf{v}_2 = 0$$

Problem Data: $D = (-15, 45) \times (-15, 15)$, $C = B_{1/2}$, $\mathrm{Re} = 200$, $T = 20$, and tangent vector

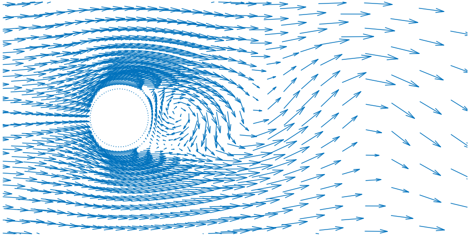
$$\tau(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (x - x_0), \quad x \in \partial C.$$

Spatial Discretization: Q2–Q1 finite elements.

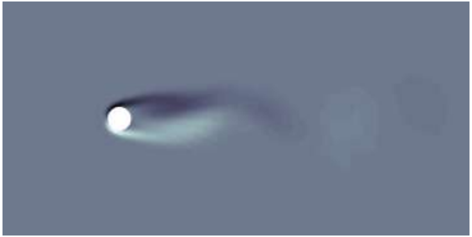
Temporal Discretization: Implicit Euler with 800 equal time steps.

Initial Vorticity ($t = 0$)

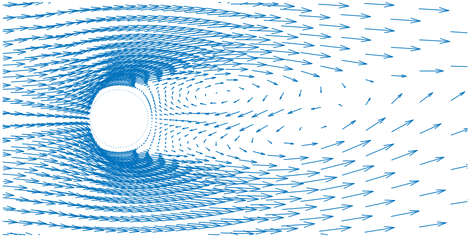
Initial Velocity ($t = 0$)



Controlled Vorticity ($t = 20$)

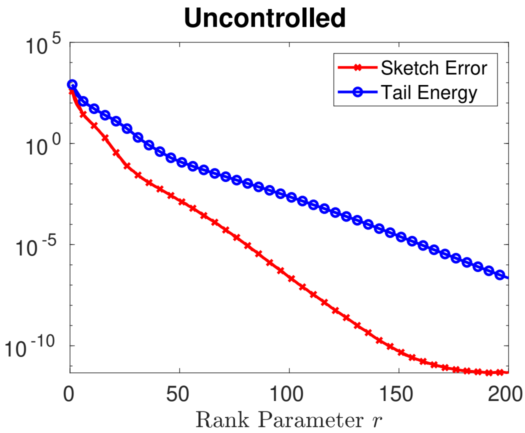


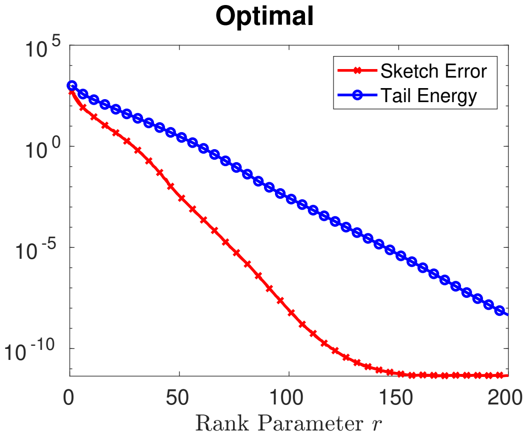
Controlled Velocity ($t = 20$)



Numerical Results

Flow Control





Sketching error averaged over 20 realizations and **tail energy** for the uncontrolled state (left) and the optimal state (right). Recall that the rank of the sketch is $k = 2r + 1$.

Numerical Results

Termination Criterion: $\|g^{(\ell)}\| \le 10^{-5}$ or $\ell > 40$.

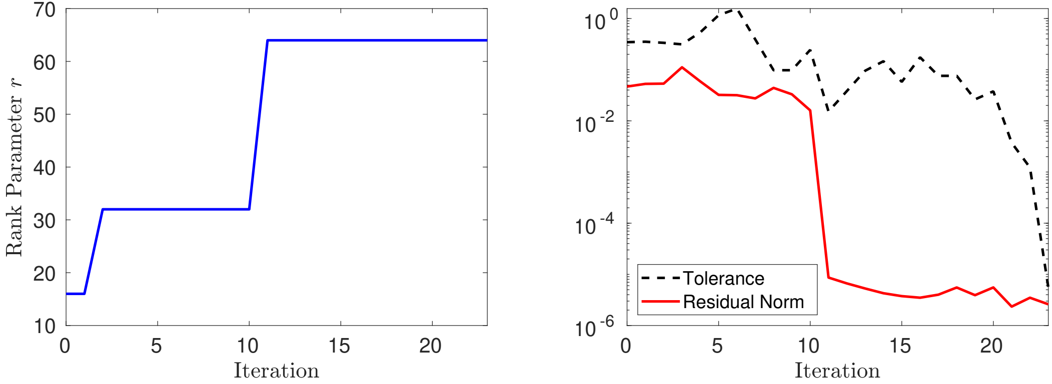
Rank Increase Function: $r \leftarrow 2r$.

rank	objective	iter	nstate	nadjoint	iterCG	compression
* 8	18.35919	40	41	15	136	45.44
*16	18.20003	40	41	33	897	23.35
*32	18.19779	40	41	31	236	11.80
64	18.19779	29	41	34	110	5.88
Adaptive	18.19779	23	27	24	121	5.88
Full	18.19779	29	30	24	107	---

*The rank 8, 16, and 32 experiments exceeded the maximum number of iterations.

Numerical Results

Flow Control



Left: **Sketch rank** as a function of iteration of adaptive-rank algorithm. **Right:** Required **gradient inexactness tolerance** and computed **residual norm** as functions of iteration.

Adpative-rank algorithm reduced memory requirement 5.88-fold!

Conclusions

- **Numerical solution** of dynamic optimization is **expensive** and requires **careful memory management**.
- We **compress** the dynamic state using **randomized matrix sketching**.
- State compression using sketching is **lossy** and results in **inexact gradients**.
- Introduced provably convergent algorithm to handle gradient inexactness. Algorithm adatively learns the rank of the optimal state — **May run out of memory for non-low-rank states!**
- Numerical examples suggest $\sim$**5–70-fold reduction** in memory requirement.

R. Muthukumar, D. P. Kouri and M. Udell, *Randomized sketching Algorithms for low-memory dynamic optimization*, 2019. `http://www.optimization-online.org/DB_FILE/2019/11/7475.pdf`