Mathematics

Spectral Problems in Inverse Scattering for Inhomogeneous Media

Fioralba Cakoni

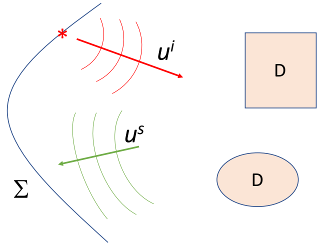
Rutgers University, Department of Mathematics

sites.math.rutgers.edu/∼fc292

Research supported by grants from AFOSR and NSF

RUTGERS

Inverse Scattering Problem



From a knowledge of u^s on Σ, for several interrogating waves u^i and possibly for a range of frequencies we can formulate two problems:

- Reconstruct everything.

  Weak scattering approximation, optimization techniques

- Obtain partial information such as the support D and estimates on the material properties.

  Qualitative approach

D. Colton and R. Kress (2019), *Inverse Acoustic and Electromagnetic Scattering Theory, 4th Edition, Springer*

Inverse Scattering

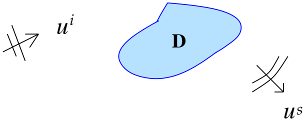
Popular approaches to the inverse scattering problem for acoustic/electromagnetic/elastic waves in the frequency domain:

1. Linearization: Ignores multiple scattering and hence model may be incorrect.
2. Nonlinear Optimization: Typically reconstruct all the unknowns. Possibly little data, but good a priori information. Convergence of Newton's Method for inverse scattering problem is not fully established. In general they do not work for anisotropic media.
3. Data Driven Models: Being developed.
4. Qualitative Method: No a priori information, but needs a lot of data. Provides partial information about the scatterer. It is mathematically rigorous with correct model.

A. Kirsch and N. Grinberg (2008), *The Factorization Method for Inverse Problems, Oxford University*.

F. Cakoni and D. Colton and H. Haddar (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publication*.

Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\Delta u^s + k^2 u^s &= 0 && \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\
\nabla \cdot A \nabla u + k^2 n u &= 0 && \text{in} \quad D \\
u &= u^s + u^i && \text{on} \quad \partial D \\
\nu \cdot A \nabla u &= \nu \cdot \nabla (u^s + u^i) && \text{on} \quad \partial D \\
\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) &= 0
\end{aligned}
$$

- k is the wave number which is proportional to the frequency ω
- u^i is the incident wave solving $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ (except for possibly one point) and u^s is the scattered wave.

$$\nabla \cdot A \nabla u^s + k^2 n u^s = \nabla \cdot (I - A) \nabla u^i + k^2 (1 - n) u^i \qquad \text{in } \mathbb{R}^3$$

where $A = I$ and $n = 1$ outside D.

Eigenvalues and Inverse Scattering Theory

Decode nonlinear information on inhomogenous media, i.e. on D, A and n, from the properties of the linear scattering operator.

Fundamental properties of this operator give rise to scattering resonances and transmission eigenvalues

- Resonances constitute a fundamental part of scattering theory. Their study has led to beautiful mathematics, has shed light into deeper understanding of direct and inverse scattering phenomena.

  R.B. MELROSE (1995), *Geometric Scattering Theory*, Cambridge University Press.

  S. DYATLOV - M. ZWORSKI (2019), *Mathematical Theory of Scattering Resonances*, AMS.

  Because the resonances are complex, it is difficult to determine them from scattering data unless they are near the real axis.

- Transmission eigenvalues are an alternative choice. They also play an intrinsic role to the scattering phenomenon.

Resonances and TEs for Spherically Stratified Media

Consider scattering of $v = j_\ell(k|x|)Y_\ell(\hat{x})$, by a ball B_1 and $n(r)$, $A = I$.

$$u^s(x) := \frac{C(k;n,\ell)}{W(k;n,\ell)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}), \quad u^\infty(x) := \frac{C(k;n,\ell)}{W(k;n,\ell)} \frac{1}{k} Y_\ell(\hat{x})$$

$$C(k;n,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1;k,n) & j_\ell(k) \\ y_\ell'(1;k,n) & kj_\ell'(k) \end{pmatrix}$$

$$W(k;n,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1;k,n) & h_\ell^{(1)}(k) \\ y_\ell'(1;k,n) & kh_\ell^{(1)'}(k) \end{pmatrix}$$

with $y_\ell(r;k,n)$ the solution (regular at $r = 0$) of

$$y'' + \frac{2}{r} + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y = 0.$$

- k such that $W(k;n,\ell) = 0$ is a scattering pole. Such k are complex with $\Im(k) < 0$.
- k such that $C(k;n,\ell) = 0$ is a transmission eigenvalue ($v = j_\ell(k|x|)Y_\ell(\hat{x})$ does not scatter). Such k can be real.

TE and Non-Scattering Frequencies

Question: Is there an incident field u^i that does not scatter?

If yes, k is such that $v := u^i|_D$ and u are solutions to the transmission eigenvalue problem

$$
\begin{aligned}
\Delta v + k^2 v = 0 & \quad \text{in} \quad D \\
\nabla \cdot A \nabla u + k^2 n u = 0 & \quad \text{in} \quad D \\
u = v & \quad \text{on} \quad \partial D \\
\nu \cdot A \nabla u = \nu \cdot \nabla v & \quad \text{on} \quad \partial D
\end{aligned}
$$

Transmission Eigenvalues

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has a non trivial solution are called transmission eigenvalues

TE and Non-Scattering Frequencies

If k is a transmission eigenvalue and the eigenfunction v that solves $\Delta v + k^2 v = 0$ in D can be extended outside D as a solution $\tilde{v}$ of the same equation, then the scattered field due to $\tilde{v}$ as an incident wave is identically zero.

In general such an extension of v does not exist (corners!).

BLÅSTEN-PÄIVÄRINTA-SYLVESTER, *Comm. Math. Phys. (2013)*

CAKONI-XIAO, *Communications in PDEs (to appear)*

Important Fact: Superposition of plane waves or point sources with source outside D, are dense in

$$\left\{ v \in H^1(D) : \; \Delta v + k^2 v = 0 \quad \text{in } D \right\}.$$

Thus at a transmission eigenvalue it is possible to superimpose plane waves to produce an arbitrarily small scattered field.

Far Field Operator & Scattering Operator

Take incident $u^i := e^{ikx\cdot\hat{d}}$

- The scattered field u^s has the asymptotic behavior

$$u^s(x;\hat{d},k) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x};\hat{d},k) + O\left(\frac{1}{|x|}\right)\right\} \qquad \text{as } |x| \to \infty$$

  uniformly in $\hat{x} = x/|x| \in \mathbb{S}^2$. u_∞ is called the far field pattern.

- We define the far field operator $F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$(F_k g)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x};\hat{d},k) g(\hat{d}) ds.$$

- F_k is related to the scattering operator $\mathcal{S}_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$\mathcal{S}_k = I + \frac{ik}{2\pi} F_k$$

Far Field Operator

Consider the incident wave-to-far field mapping

$$\mathbb{G} : v \in \left\{H^1(D) : \Delta v + k^2 v = 0\right\} \mapsto u_\infty(\hat{x}) \in L^2(\mathbb{S}^2)$$

where u_∞ is the far field pattern of the scattered field u^s satisfying

$$\nabla \cdot A\nabla u^s + k^2 n u^s = \nabla \cdot (I - A)\nabla v + k^2(1 - n)v \qquad \text{in } \mathbb{R}^3$$

$$F_k g = \mathbb{G}(v_g)$$

where the incident wave is a Herglotz wave function

$$v_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot\hat{d}} g(\hat{d}) ds.$$

Injectivity of the Far Field Operator

Theorem

$\mathbb{G} : L^2(D) \to L^2(\mathbb{S}^2)$ is not injective if and only if k is a transmission eigenvalue. In this case

$$\mathbb{G}v = 0$$

where v is part of the eigenfunction solving the Helmholtz equation.

Theorem

$F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is not injective if and only if

1. k is a transmission eigenvalue and
2. the component v that satisfies $\Delta v + k^2 v = 0$ of the corresponding eigenfunction is a Herglotz wave function

$$v_g(x) := \int_{\mathbb{S}^2} e^{ikx\cdot\hat{d}} g(\hat{d}) ds.$$

The Range of Far Field Operator

Is $\quad e^{-ik\hat{x}\cdot z} \in \text{Range}(F_k)$?

Note that $e^{-ik\hat{x}\cdot z}$ is the far field pattern of $\Phi(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$.

$$e^{-ik\hat{x}\cdot z} \in \text{Range}(\mathbb{G}) \quad \Longleftrightarrow \quad z \in D$$

In particular $\mathbb{G}(v_z) = e^{-ik\hat{x}\cdot z}$ where

$$\begin{aligned}
\Delta v_z + k^2 v_z &= 0 && \text{in} \quad D \\
\nabla \cdot A \nabla u_z + k^2 n u_z &= 0 && \text{in} \quad D \\
u_z - v_z &= \Phi(\cdot, z) && \text{on} \quad \partial D \\
\nu \cdot A \nabla u_z - \nu \cdot \nabla v_z &= \nu \cdot \nabla \Phi(\cdot, z) && \text{on} \quad \partial D
\end{aligned}$$

Generalized linear sampling method

For $\alpha > 0$, consider $J_\alpha(g) = \alpha \mathbb{B}(g) + \|F_k g - \varphi\|^2_{L^2(\mathbb{S}^2)}$

where $\mathbb{B} : L^2(\mathbb{S}^2) \to \mathbb{R}^+$ is a functional (not necessarily convex).

Let g^z_α be a minimizing sequence of J_α such that

$$J_\alpha(g^z_\alpha) \leq \inf_{g \in L^2(\mathbb{S}^2)} J_\alpha(g) + o(\alpha).$$

Then

$$\varphi \in \text{Range}(\mathbb{G}) \iff \lim_{\alpha \to 0} \mathbb{B}(g_\alpha) < \infty$$

Key in the choice of $\mathbb{B}(g)$ are

- $\mathbb{B}(g)$ must be bounded if and only if $\|v_g|_D\|_{H^1(D)}$ is bounded.
- $\mathbb{B}(g)$ must be expressed in terms of F_k (i.e. the data).

AUDIBERT - HADDAR (2015) Ph.D Thesis, Ecole Polytechnique.

Generalized Linear Sampling Method

Main assumption: A, n are such that the interior transmission problem is Fredholm of index one, i.e. a compact perturbation of an invertible operator.

$$F_k = H^* \mathbb{T} H, \qquad Hg := v_g|_D$$

A possible choice of $\mathbb{B}$ is

$$F_k^{\#} := |\Re(F_k)| + |\Im(F_k)|$$

hence using the functional

$$J_\alpha(g) = \alpha \left| \left(F_k^{\#} g, g \right) \right| + \left\| F_k g - e^{-ik\hat{x}\cdot z} \right\|^2_{L^2(\mathbb{S}^2)}$$

A. Kirsch and N. Grinberg (2008), *The Factorization Method for Inverse Problems, Oxford University*.

The Determination of the Support D

Theorem

Assume $k > 0$ is not a transmission eigenvalue. Then

$$z \in \mathbb{R}^3 \setminus \overline{D} \qquad \Longleftrightarrow \qquad \lim_{\alpha \to 0} \left| \left(F_k^{\#} g_\alpha^z, g_\alpha^z \right) \right| = +\infty$$

$\left| \left(F_k^{\#} g_\alpha^z, g_\alpha^z \right) \right|$ is an indicator function for the support D

F. Cakoni and D. Colton and H. Haddar (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publication.*

Determination of **Real** Transmission Eigenvalues

Theorem

$k > 0$ is a transmission eigenvalue

$$\Longleftrightarrow$$

$$\lim_{\alpha \to 0} \left| \left(F_k^{\#} g_\alpha^z, g_\alpha^z \right) \right| = +\infty \text{ for all } z \in B_\delta \subset D$$

$\left| \left(F_k^{\#} g_\alpha^z, g_\alpha^z \right) \right|$ determines the real transmission eigenvalues

F. Cakoni and D. Colton and H. Haddar (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publication*.

Qualitative Methods

Without any a priori information on the constitutive material properties or the topology of scatterers

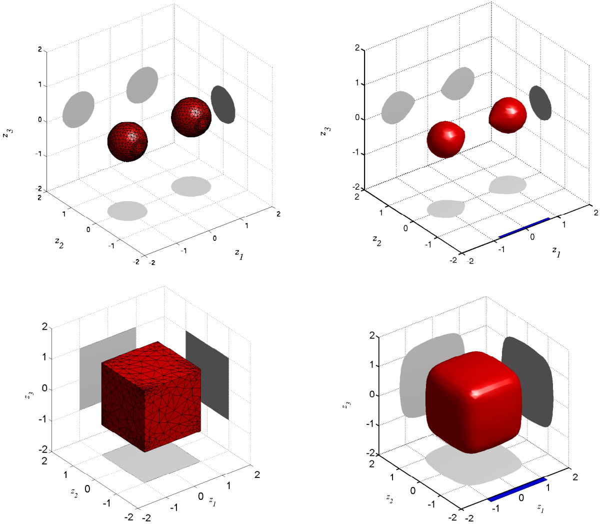
A knowledge of the far field operator

$$(F_k g)(\hat{x}) := \int_{\mathbb{S}_0} u_\infty(\hat{x}; \hat{d}, k) g(\hat{d}) ds$$

$\hat{x}$, $\hat{d} \in \mathbb{S}_0 \subset \mathbb{S}$, $z \in \mathbb{R}^3$, $k \in [k_0, k_1]$, determines:

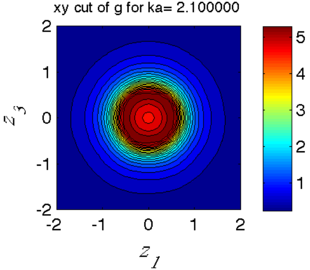
- The support D
- Real transmission eigenvalues

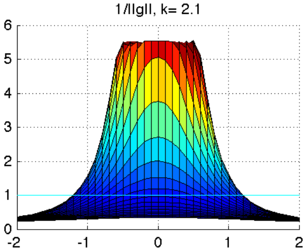
Shape Reconstruction

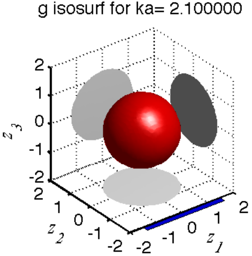


F. CAKONI AND D. COLTON AND P. MONK (2011), *The Linear Sampling Method in Inverse Electromagnetic Scattering, CBMS-NSF, SIAM Publication.*

Examples of Reconstruction

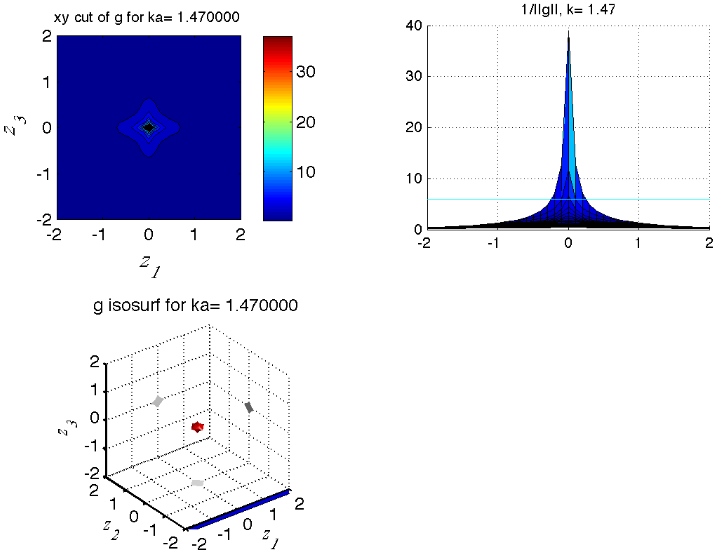






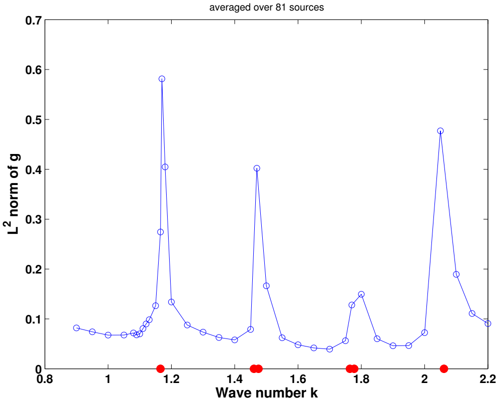
$D := B_1$, $A = I, n = 16$, k is not a TE

Examples of Reconstruction



$D := B_1$, $A = I, n = 16$, k is a TE

Computation of Real Transmission Eigenvalues



The peaks correspond to the computed transmission eigenvalues.
Red dots indicate exact transmission eigenvalues.

Transmission Eigenvalue Problem

Having determined the support D and (a few) transmission eigenvalues, the aim is to get some information about the constitutive parameters A and n.

For this we appeal to the transmission eigenvalue problem:

$$\begin{aligned} \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ \nabla \cdot A \nabla w + k^2 n w = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \nu \cdot A \nabla w = \nu \cdot \nabla v \quad & \text{on} \quad \partial D \end{aligned}$$

$$\int_D A \nabla w \cdot \nabla \overline{w}' \, dx - \int_D \nabla v \cdot \nabla \overline{v}' \, dx - k^2 \int_D n w \, \overline{w}' \, dx + k^2 \int_D v \, \overline{v}' \, dx = 0$$

Non-elliptic in the sense of Agmon-Douglis-Nirenberg!

State of the Art Results on TEP

The transmission eigenvalue problem is non-self adjoint!

- When $\Im(A) < 0$ or/and $\Im(n) > 0$ all transmission eigenvalues are complex.
- Spectral analysis and completeness results under the assumptions A, n are C^1 only near ∂D (otherwise L^∞), real-valued, and satisfy the complementing boundary conditions,

$$\langle A\nu,\nu\rangle\langle A\xi,\xi\rangle - \langle A\nu,\xi\rangle^2 \neq 1 \quad \text{on } \partial D$$

$$\langle A\nu,\nu\rangle n \neq 1 \quad \text{on } \partial D$$

  for all $\xi\cdot\nu = 0$ where ν is the normal vector to ∂D

- Location of transmission eigenvalues in the complex plane for A scalar function, and A, n both $C^\infty(D)$ in addition to above condition.

CAKONI-GINTIDES-HADDAR (2010), CAKONI-KIRSCH (2010), BONNET-BEN DHIA-CHESNEL-HADDAR (2011),

SYLVESTER (2012), LEUNG-COLTON (2012-13-15), HM. NGUYEN-QH. NGUYEN (2016), ROBIANNO (2013),

LAKSHTANOV-VAINBERG (2015-16), VODEV (2017-18), HM. NGUYEN (2020) ETC....

Monotonicity property of TEs

If $n-1$, $A-I$ have the same sign in D then there exists a sequence of real eigenvalues $k_j(A, n, D)$ accumulating to $+\infty$ and they satisfy monotonicity properties it terms of A, n and D. For example for $B_2 \subset D \subset B_1$

- If $n > 1$ and $A = I$ in D

$$k_j(n_{max}, B_1) \le k_j(n_{max}, D) \le k_j(n(x), D) \le k_j(n_{min}, D) \le k_j(n_{min}, B_2)$$

- If $A < 1$ and $n = I$ in D

$$k_j(a_{min}, B_1) \le k_j(a_{min}, D) \le k_j(A(x), D) \le k_j(a_{max}, D) \le k_j(a_{max}, B_2)$$

CAKONI-GINTIDES-HADDAR (2010), CAKONI-KIRSCH (2010)

Numerical Example: Anisotropic Media

For a given (unknown) anisotropic media A, we find an isotropic media a_0 that has the first transmission eigenvalue the same as the (measured) first transmission eigenvalue for anisotropic media. Monotonicity gives that this a_0 is between a^* and a_*.

We consider D to be the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$, $n = 1$

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

Matrix	Eigenvalues a_*, a^*	Predicted a_0
A_{iso}	4, 4	4.032
A_1	2, 8	5.319
A_2	6, 8	7.407
A_{2r}	6, 8	6.896

F. Cakoni, D. Colton, P. Monk, J. Sun (2010) The inverse electromagnetic scattering problem for anisotropic media, *Inverse Problems*.

TE and Non-desctructive Testing

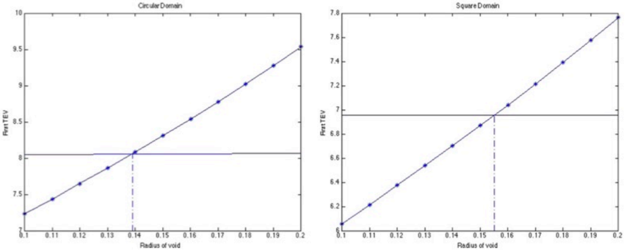


Figure 6. The first TEV of the two domains with a single void versus the radius of the void. The horizontal lines are the k_1's for the two domains with two voids. The vertical dotted lines are the approximated values of ϵ^* such that a void of the form $B_{\epsilon^*}(0, 0)$ gives the same TEV approximately, i.e. $k_1(\text{void(s)}) \approx k_1(B_{\epsilon^*}(0, 0))$.

Table 6. Qualitative reconstruction of area from FF-measurements.

D	D_1	$\lvert B_{\epsilon^*}(0, 0)\rvert$	$\lvert D_1\rvert$
Disk $R = 1$	Disk $r = 0.1$	0.0328	0.0314
	Square	0.0303	0.0300
$[-1, 1] \times [-1, 1]$	Ellipse	0.0613	0.0628
	Square	0.0749	0.1256

F. Cakoni, I. Harris, and J. Sun (2014) Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids, *Inverse Problems*.

Cons of Using Transmission Eigenvalues

Drawbacks of using the transmission eigenvalues as target signatures

- The method of transmission eigenvalues does not apply to absorbing or dispersive media.
- Multifrequency data is needed. The first transmission eigenvalue is determined by the material properties of the scatterer, i.e. one can not choose the range of interrogating frequencies.

It is possible to modify the scattering data in such away that the same analysis yields a new eigenvalue problem whose eigenvalue parameter is not related to the interrogating frequency

$$u^{total} = u^{scattered} + u^{incident}$$

u^{total} is the physical field, thus change $u^{scattered}$ by changing $u^{incident}$.

Modified Far-Field Operator

Let $h^{s,\lambda}$ be the scattered field due a plane wave by an artificial scatterer depending on $\lambda \in \mathbb{C}$, and let h_∞^λ be its far field. Define

$$\mathcal{F}g = F_k g - F_\lambda g := \int_{\mathbb{S}^2} \left[u_\infty(\cdot\,;\hat{d}) - h_\infty^\lambda(\cdot\,;\hat{d}) \right] g(\hat{d}) ds$$

- F_k is known from measurements.
- F_λ is computed (can be precomputed).

If the above analysis on F_k is now performed to the modified far field operator $\mathcal{F}$, a new eigenvalue problem appears instead of the transmission eigenvalue problem.

Eigenvalues and Inverse Scattering Theory

Example (Steklov Eigenvalues)

$F_\lambda : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by

$$(F_\lambda)g = \int_{\mathbb{S}} h_\infty^\lambda(\hat{x}, d)g(d)\, ds(d)$$

where h_∞^λ is the far field pattern of the scattered field $h^s := h^{s,\lambda}$

$$\begin{aligned} \Delta h^s + k^2 h^s &= 0 && \text{in} \quad \mathbb{R}^3 \setminus \overline{D_b} \\ h &= h^s + e^{ik\hat{d}\cdot x} && \\ \frac{\partial h}{\partial \nu} + \lambda h &= 0 && \text{in} \quad \partial D_b \\ \lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) &= 0 && r = |x| \end{aligned}$$

and D_b is such that $D \subseteq D_b$.

Steklov Eigenvalues

The question if there is a $g \in L^2(\mathbb{S}^2)$ s.th. $\mathcal{F}g = F_k g - F_\lambda g = 0$ yield

$$\begin{aligned} \nabla \cdot A \nabla u + k^2 n u &= 0 \qquad \text{in } D_b \\ \nu \cdot A \nabla u + \lambda u &= 0 \qquad \text{on } \partial D_b. \end{aligned}$$

with $A = I$, $n = 1$ in $D_b \setminus D$. If k is fixed then this is the Steklov eigenvalue problem for λ

Theorem (Audibert-Cakoni-Haddar, IP 2017)

$$\lambda \in \mathbb{C} \text{ is a Steklov eigenvalue} \quad \Longleftrightarrow$$

$$\lim_{\alpha \to 0} |(F_\lambda g_\alpha^z, g_\alpha^z)| = +\infty \text{ for all } z \in B_\delta \subset D_b$$

where g_α^z is a minimizing sequence of

$$J_\alpha(g) = \alpha \, |(F_\lambda g, g)| + \|\mathcal{F}g - e^{-ik\hat{x}\cdot z}\|^2_{L^2(\mathbb{S}^2)}$$

Modified Far Field Operator

Example (New eigenvalue problems)

$F_\lambda : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is defined by

$$F_\lambda g = \int_S h_\infty^\lambda(\hat{x}, d) g(d)\, ds(d)$$

where h_∞^λ is the far field pattern of the scattered field $h^s := h^{s,\lambda}$

$$\begin{aligned}
\Delta h^s + k^2 h^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D_b} \\
a\Delta h + k^2 \lambda h &= 0 && \text{in } D_b \\
h &= h^s + e^{ik\hat{d}\cdot x} && \text{on } \partial D_b \\
a\frac{\partial h}{\partial \nu} &= \frac{\partial (h^s + e^{ik\hat{d}\cdot x})}{\partial \nu} && \text{on } \partial D_b \\
\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) &= 0 && r = |x|
\end{aligned}$$

The chosen parameter a is a fixed. The only requirement is that the above scattering problem is well-posed.

Eigenvalues and Inverse Scattering Theory

The question if there is a $g \in L^2(\mathbb{S}^2)$ s.th. $\mathcal{F}g = Fg - F_\lambda g = 0$ yield the following modified transmission eigenvalue problems

$$
\begin{aligned}
\nabla \cdot A\nabla u + k^2 n u = 0 \qquad & \text{in} \qquad D_b \\
a\Delta v + k^2 \lambda v = 0 \qquad & \text{in} \qquad D_b \\
u = v \qquad & \text{on} \qquad \partial D_b \\
\nu \cdot A\nabla u = \nu \cdot a\nabla v \qquad & \text{on} \qquad \partial D_b
\end{aligned}
$$

with $A = I$, $n = 1$ in $D_b \setminus D$. With fixed $a > 0$ this modification was first introduced in

S. Cogar, D. Colton, S. Meng and P. Monk (2017) The modified transmission eigenvalue problem in inverse scattering, *Inverse Problems*.

Eigenvalues and Inverse Scattering Theory

$$\nabla \cdot A\nabla u + k^2 n u = 0 \quad \text{in} \quad D_b$$
$$a\Delta v + k^2 \lambda v = 0 \quad \text{in} \quad D_b$$
$$u = v \quad \text{on} \quad \partial D_b$$
$$\nu \cdot A\nabla u = \nu \cdot a\nabla v \quad \text{on} \quad \partial D_b$$

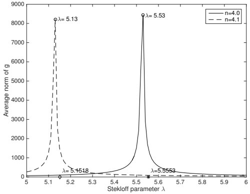
The eigenvalues $\lambda \in \mathbb{C}$ can be computed using data at fixed frequency k by a similar approach as the other type of eigenvalues.

Negative Refractive Index (Audibert-Cakoni-Haddar 2017, Cakoni-Levitin 2020)

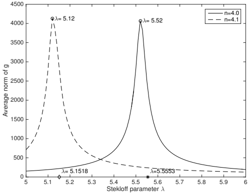
Fix an interrogating frequency k, and choose $-1 \neq a < 0$.
If A and n are real valued, this is an eigenvalue problem for a selfadjoint compact operator (not necessary sign-definite) with eigen-parameter λ.

Steklov Eigenvalues

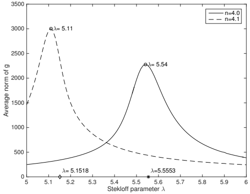
D unit disk, $k = 1$, $n(x)$ changes form 4 to 4.1, 51 direction all around.



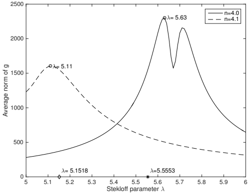
$\epsilon = 0 (0\%)$



$\epsilon = 0.01 (.57\%)$



$\epsilon = 0.05 (2.9\%)$

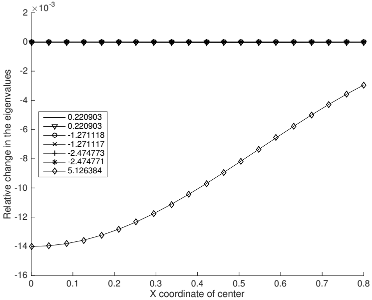


$\epsilon = 0.15 (8.6\%)$

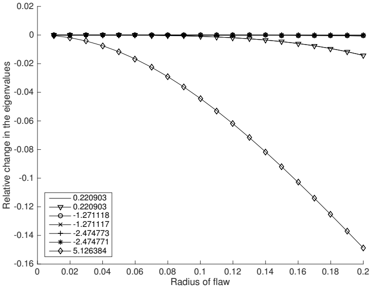
Noise added point-wise. Percentage is the relative ℓ^2 norm

Sensitivity of Eigenvalues: Unit Disk with Flaw

The “flaw” is a circular region of radius r_c centered at $(x_c, 0)$ with $n(x) = 1$ inside the flaw. Noise $\epsilon = 0.01$. Wavenumber $k = 1$.



Changing x_c, $r_c = 0.05$

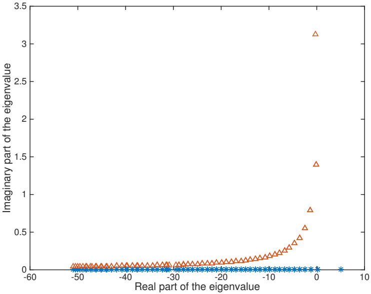


Changing r_c, $x_c = 0.3$.

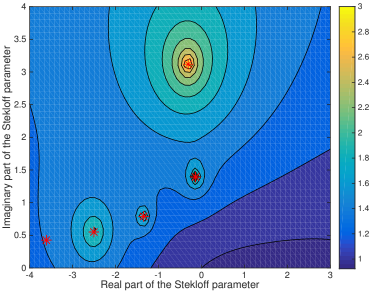
Plot $(\lambda^c_{j^*} - \lambda_j)/|\lambda_j|$, $j = 1, \cdots, 7$ against geometric parameters.

Complex Eigenvalues: Unit Disk $n(x) = 4 + 4i$

Complex eigenvalues can be detected by the same procedure as before but now searching in a region in the complex plane.



Comparison of eigenvalues for $n(x) = 4$ (blue) and $n(x) = 4 + 4i$ (red)



Contours of $\log_{10}(\|g\|)$. Exact Steklov eigenvalues are shown as *.

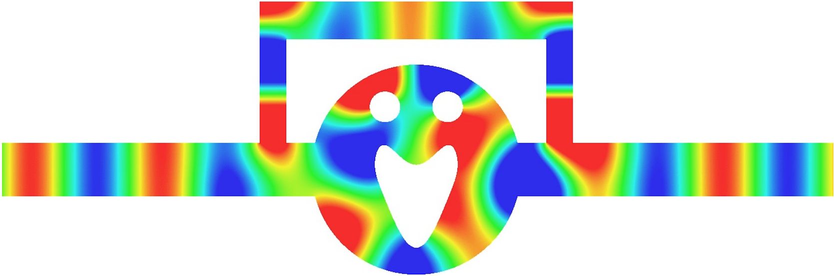
F. Cakoni, D. Colton, S. Meng and P. Monk (2016) Steklov eigenvalues in inverse scattering, *SIAM J. Appl. Math*.

New Eigenvalues Problems

- We introduced a general framework of modifying the far field operator involving a family of computable scattering problems.
- The injectivity of the modified far field operator relates to the existence of non-trivial solutions to a family of homogeneous problems, hence leading to new eigenvalue problems.
- A broad question is how to design modifications of the data operator, in other words define artificial background scattering problems appropriate to a specific application.

Nonreflected, Nontransmitted Modes in Waveguides

Thanks to Luca Chesnel, CMAP [Click]



A-S Bonnet-Ben Dhia, L. Chesnel and V. Pagneux (2018), Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem, *Proc. R. Soc. A*.

Transmission Eigenvalues and the Riemann Hypothesis

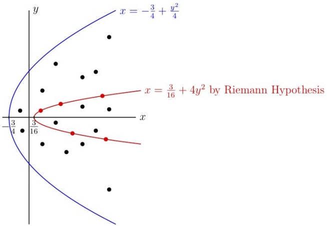
The concept of transmission eigenvalues can also be considered in connection with scattering theory in hyperbolic geometry, particularly for automorphic solutions of the wave equation in the hyperbolic plane with isometries corresponding to the discrete groups.

In this context, scattering results from the interaction of an incident field with the boundary of the fundamental domain.

For more details see

F. CAKONI AND S. CHANILLO (2019) Transmission eigenvalues and the Riemann zeta function in scattering theory for automorphic forms on Fuchsian groups of type I.

Transmission Eigenvalues and the Riemann Hypothesis



Theorem (Cakoni-Chanillo, 2019)

In the above context, the Riemann hypothesis is equivalent to the statement that all transmission eigenvalues lie on the parabola

$$x = 3/16 + 4y^2$$

except for the trivial eigenvalues $\lambda = 0$ and $\lambda = 1/4$

Literature

F. Cakoni and D. Colton and H. Haddar (2016), Inverse Scattering Theory and Transmission Eigenvalues, *CBMS-NSF, SIAM Publication*.

F. Cakoni, D. Colton, S. Meng and P. Monk (2016) Steklov eigenvalues in inverse scattering, *SIAM J. Appl. Math*.

L. Audibert, F. Cakoni and H. Haddar (2017) New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *Inverse Problems*.

S. Cogar, D. Colton, S. Meng and P. Monk (2017) The modified transmission eigenvalue problem in inverse scattering, *Inverse Problems*.

L. Audibert, L. Chesnel and H. Haddar (2018) Transmission eigenvalues with artificial background for explicit material index identification, *C. R. Acad. Sci. Paris, Ser. I*.

S. Cogar (2020), Analysis of a trace class Stekloff eigenvalue problem arising in inverse scattering, *SIAM J. Applied Mathematics*.

A-S Bonnet-Ben Dhia, L. Chesnel and V. Pagneux (2018), Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem, *Proc. R. Soc. A*.

F. Cakoni and S. Chanillo (2019), Transmission eigenvalues and the Riemann zeta function in scattering theory for automorphic forms on Fuchsian groups of type I., *Acta Math Sinica*.