Center for Mathematics and Artificial Intelligence

PDE Constrained Optimization: Implementation Aspects

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May 02, 2022

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Outline

PDE Constrained Optimization – Overview

Derivative Computation

Application of Abstract Approach

Finite Element Method

Matlab Code: Newton CG, L-BFGS, Semismooth Newton

PDE Constrained Optimization (PDECO) Overview

Literature: [Lions, 1971, Tröltzsch, 2010, Ito and Kunisch, 2008, Antil et al., 2018, Hinze et al., 2009, Ulbrich, 2011, Attouch et al., 2006, Ekeland and Témam, 1999]

PDE constrained optimization has 4-5 major components:

- **control** z
- **state** u
- **state equation** (i.e., PDE) which associates with every control z a state u.
- **cost functional** J which depends on z and u, to be minimized;
- constraints obeyed by z (**control constraints**) and u (**state constraints**);

In the most abstract form this problem can be written as

$$\begin{aligned}
&\min J(u,z) && \\
&\text{s.t. } e(u,z) = 0, && \text{PDE} \\
&z \in Z_{ad}, && \text{control constraints} \\
&u \in U_{ad}, && \text{state constraints}
\end{aligned}$$

Abstract PDECO Problem

$$
\begin{aligned}
&\min J(u,z) && \\
&\text{s.t. } e(u,z) = 0, && \text{PDE} \\
&z \in Z_{ad}, && \text{control constraints} \\
&u \in U_{ad}, && \text{state constraints}
\end{aligned}
$$

where u: states, z: controls,

- $(U, \|\cdot\|_U)$, $(Y, \|\cdot\|_Y)$ Banach spaces, $(Z, \|\cdot\|_Z)$ is a reflexive Banach space.
- $Z_{ad} \subset Z$ and $U_{ad} \subset U$ are nonempty, closed convex set.
- $J : U \times Z \to \mathbb{R}$, $e : U \times Z_{ad} \to Y$ are sufficiently smooth mappings.
- Can think $(U, \|\cdot\|_U) = (\mathbb{R}^n, \|\cdot\|_2)$, $(Y, \|\cdot\|_Y) = (\mathbb{R}^n, \|\cdot\|_2)$, $(Z, \|\cdot\|_Z) = (\mathbb{R}^m, \|\cdot\|_2)$.

For simplicity we shall focus on the control constrained case, i.e., $U_{ad} = U$.

Three different formulations

- **Full space form**

$$
\begin{aligned}
&\min J(u,z) && \\
&\text{s.t. } e(u,z) = 0, && \text{PDE} \\
&z \in Z_{ad}, && \text{control constraints}
\end{aligned}
$$

- Let $x = (u,z)$, $X = U \times Z$, $X_{ad} = U \times Z_{ad}$ then we arrive at the following abstract problem

$$
\begin{aligned}
&\min J(x) && \\
&\text{s.t. } e(x) = 0, && \text{PDE} \\
&x \in X_{ad}. && \text{constraints}
\end{aligned}
$$

- **Reduced form.** Let us consider a map

$$S : \mathcal{Z} \to U, \quad z \mapsto S(z) = u \text{ with } e(S(z),z) = 0.$$

Here $\mathcal{Z}$ is open such that $Z_{ad} \subset \mathcal{Z} \subset Z$. We assume that for $z \in \mathcal{Z}$ there is a unique $u = S(z) \in U$. Then we can write

$$\min_{z \in Z_{ad}} \mathcal{J}(z) := J(S(z),z)$$

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Derivative Computation

Compute gradient and Hessian information for

$$\mathcal{J}(z) = J(S(z), z), \quad \text{where } S(z) = u \text{ solves } e(u, z) = 0$$

Assumption

- J and e are twice continuously differentiable,
- $e_u(u, z)$ is continuously invertible.

Use adjoint equation approach for derivative computation.

See, e.g., Chapter 1 in [Hinze et al., 2009], Chapter 1 in [Antil et al., 2018] or [Heinkenschloss, 2008].

Gradient Computation: Reduced Form

- Derivative

$$\langle \mathcal{J}'(z), h \rangle_{Z^*,Z} = \langle J_u(S(z), z), S'(z)h \rangle_{U^*,U} + \langle J_z(S(z), z), h \rangle_{Z^*,Z}$$

- Implicit function theorem applied to $e(S(z), z) = 0$ gives

$$\begin{aligned} e_u(S(z), z)\left(S'(z)h\right) + e_z(S(z), z)h = 0 \\ \Longrightarrow S'(z)h = -e_u^{-1}(S(z), z)e_z(S(z), z). \end{aligned}$$

- Derivative $(u = S(z))$

$$\begin{aligned} &\langle \mathcal{J}'(z), h \rangle_{Z^*,Z} \\ &= \langle J_u(u, z), -e_u(u, z)^{-1} e_z(u, z)h \rangle_{U^*,U} + \langle J_z(u, z), h \rangle_{Z^*,Z} \\ &= \langle \underbrace{e_u(u, z)^{-*} J_u(u, z)}_{=:p}, -e_z(u, z)h \rangle_{Y^*,Y} + \langle J_z(u, z), h \rangle_{Z^*,Z} \\ &= \langle -e_z(u, z)^* p + J_z(u, z), h \rangle_{Z^*,Z} \end{aligned}$$

- Connection with Lagrangian $L(u,z,p) = J(u,z) - \langle p, e(u,z)\rangle_{Y^*,Y}$:
  - The adjoint variable p solves $e_u^*(u,z)p = J_u(u,z)$,
    which is equivalent to $L_u(u,z,p) = 0$.
  - Derivative

$$
\begin{aligned}
\langle \mathcal{J}'(z), h\rangle_{Z^*,Z} &= \langle -e_z(u,z)^* p + J_z(u,z), h\rangle_{Z^*,Z} \\
&= \langle L_z(u,z,p), h\rangle_{Z^*,Z}
\end{aligned}
$$

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- Assuming that Z is a Hilbert space the **gradient** $\nabla\mathcal{J}(z)$ is such that

$$\begin{aligned}(\nabla\mathcal{J}(z), h)_Z &= \langle \mathcal{J}'(z), h\rangle_{Z^*,Z} \\ &= \langle -e_z(u,z)^* p + J_z(u,z), h\rangle_{Z^*,Z} \quad \forall h \in Z\end{aligned}$$

(Riesz representation Theorem)

- Connection with Lagrangian $L(u,z,p) = J(u,z) - \langle p, e(u,z)\rangle_{Y^*,Y}$:
  - The adjoint variable p solves $e_u^*(u,z)p = J_u(u,z)$,
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\begin{aligned}
(\nabla\mathcal{J}(z), h)_Z &= \langle \mathcal{J}'(z), h\rangle_{Z^*,Z} \\
&= \langle -e_z(u,z)^* p + J_z(u,z), h\rangle_{Z^*,Z} \quad \forall h \in Z
\end{aligned}
$$

  (Riesz representation Theorem)

Derivative Computation Using Adjoints

1. Given z, solve $e(u,z) = 0$ for u (if not done already).
2. Solve the adjoint equation $e_u(u(z),z)^* p = J_u(u(z),z)$ for p . Denote the solution by $p(z)$.
3. Compute $\mathcal{J}'(z) = J_z(u(z),z) - e_z(u(z),z)^* p(z)$.

Two PDE solves (possibly nonlinear in step 1, linear PDE in step 2)

Hessian Computation

- Apply **implicit differentiation** to

$$\mathcal{J}'(z) = J_z(u(z), z) - e_z(u(z), z)^* p(z)$$

to compute Hessian information.

Hessian Computation

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- This leads to

$$\mathcal{J}''(z) = T(S(z), z)^* H(S(z), z, p) T(S(z), z)$$

  where

$$T(u, z) = \begin{pmatrix} -e_u(u, z)^{-1} e_z(u, z) \\ I_Z \end{pmatrix}$$

  and $H(u, z, p)$ is given by

$$H(u, z, p) = \begin{pmatrix} L_{uu}(u, z, p) & L_{uz}(u, z, p) \\ L_{zu}(u, z, p) & L_{zz}(u, z, p) \end{pmatrix}$$

Hessian Computation

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$$H(u, z, p) = \begin{pmatrix} L_{uu}(u, z, p) & L_{uz}(u, z, p) \\ L_{zu}(u, z, p) & L_{zz}(u, z, p) \end{pmatrix}$$
- In practice we are interested in computing **Hessian times vector**. For instance in the absence of control constraints we need to solve
$$\mathcal{J}'(z) = 0$$
which we can do so by solving Newton's equation for s_z in an iterative fashion
$$\mathcal{J}''(z) s_z = -\nabla \mathcal{J}(z).$$

Hessian-Times-Vector Computation

- **Hessian-Times-Vector Computation**
  1. Given z, solve $e(u,z)=0$ for u (if not done already).
  2. Solve adjoint equation: $e_u(u,z)^* p = J_u(u,z)$ for p (if not done already).
  3. Solve $e_u(u,z)w = -e_z(u,z)v$.
  4. Solve $e_u(u,z)^* q = D_{uu}L(u,z,p)w + D_{uz}L(u,z,p)v$.
  5. Compute
  $$\mathcal{J}''(z)v = -e_z(u,z)^* q + L_{uz}(u,z,p)w + L_{zz}(u,z,p)v.$$

  Two linear PDE solves in steps 3+4 per direction v.

- Vector s_z solves Newton equation $\mathcal{J}''(z)s_z = -\mathcal{J}'(z)$ if and only if (s_u, s_z) solves the quadratic program

$$\begin{aligned}
\min \quad & \left\langle \begin{bmatrix} J_u(\cdot) \\ J_z(\cdot) \end{bmatrix}, \begin{bmatrix} s_u(\cdot) \\ s_z(\cdot) \end{bmatrix} \right\rangle + \frac{1}{2}\left\langle \begin{bmatrix} s_u(\cdot) \\ s_z(\cdot) \end{bmatrix}, \begin{bmatrix} L_{uu}(\cdot\cdot) & L_{uz}(\cdot\cdot) \\ L_{zu}(\cdot\cdot) & L_{zz}(\cdot\cdot) \end{bmatrix} \begin{bmatrix} s_u(\cdot) \\ s_z(\cdot) \end{bmatrix} \right\rangle \\
\text{s.t.} \quad & e_u(\cdot)s_u + e_z(\cdot)s_z = 0,
\end{aligned}$$

where $(\cdot) = (u(z), z)$ and $(\cdot\cdot) = (u(z), z, p(z))$.

Outline

Semilinear Elliptic PDECO Problem

- Problem:

$$\text{Minimize } J(u,z) = \frac{1}{2}\int_\Omega |u(x) - u_d(x)|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |z(x)|^2 \,\mathrm{d}x$$

s.t.

$$\begin{aligned} -\mathrm{div}(A\nabla u) + f(\cdot, u) &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$$z \in Z_{ad} := \{z \in L^\infty(\Omega) \ : \ z_a(x) \le z(x) \le z_b(x) \quad \text{a.e. in } \Omega\}$$

with $z_a, z_b \in L^\infty(\Omega)$ and $z_a < z_b$ a.e. in Ω.

- **Control space:** $L^\infty(\Omega)$. For existence we use $L^p(\Omega)$ with $N/2 < p < \infty$. Notice that $Z_{ad} \subset L^\infty(\Omega) \subset L^p(\Omega)$.

Semilinear Elliptic PDECO Problem

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- **Control space:** $L^\infty(\Omega)$. For existence we use $L^p(\Omega)$ with $N/2 < p < \infty$. Notice that $Z_{ad} \subset L^\infty(\Omega) \subset L^p(\Omega)$.
  State space: $U = L^\infty(\Omega) \cap H_0^1(\Omega)$. [Antil et al., 2017, Antil and Warma, 2020].

Semilinear Elliptic PDECO Problem

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State space: $U = L^\infty(\Omega) \cap H_0^1(\Omega)$. [Antil et al., 2017, Antil and Warma, 2020].
- Notice that $Z \subset L^\infty(\Omega)$. This is a sufficient condition for the differentiability of S, a Nemytskii (superposition) operator.

- Lagrangian:

$$L(y,u,p) = \frac{1}{2}\int_\Omega |u(x)-u_d(x)|^2\,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |z(x)|^2\,\mathrm{d}x \\ - \int_\Omega \big(A\nabla u\cdot\nabla p + f(x,u)p - zp\big)\,\mathrm{d}x.$$

- Adjoint equation:

$$\begin{aligned} -\mathrm{div}(A^T\nabla p) + f_u(\cdot,u)p &= (u-u_d) \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Derivative

$$\langle \mathcal{J}'(z), h\rangle_{Z^*,Z} = \langle \lambda z + p, h\rangle_{Z^*,Z}.$$

- Gradient: $\nabla\mathcal{J}$ is computed as follows:

$$(\nabla\mathcal{J}(z), h)_Z = \langle \mathcal{J}'(z), h\rangle_{Z^*,Z}, \quad \forall h\in Z.$$

- Hessian: $L_{uz} = L_{zu} = 0$. Then use the approach described before.

Linear Parabolic PDECO Problem

- Problem:

$$\text{Minimize } \frac{1}{2}\int_0^T \int_\Omega |u(x,t) - u_d(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\lambda}{2}\int_0^T \int_\Omega |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t$$

where u solves

$$\begin{aligned}
\partial_t u(x,t) - \text{div }(A\nabla u) + \mathbf{b}\cdot\nabla u &= z \quad \text{in } \Omega \times (0,T) \\
u &= 0, \quad \text{on } \Gamma_D \times (0,T), \\
A\nabla u \cdot \boldsymbol{\nu} &= 0 \quad \text{on } \Gamma_N \times (0,T) \\
u(x,0) &= 0 \quad \text{in } \Omega.
\end{aligned}$$

- Lagrangian (formally)

$$\begin{aligned}
L(u,z,p) =& \frac{1}{2}\int_0^T \int_\Omega |u(x,t) - u_d(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\lambda}{2}\int_\Omega |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
& \int_0^T \int_\Omega \big(\partial_t u(x,t)p - \text{div }(A\nabla u)p + \mathbf{b}\cdot\nabla u p - zp\big) \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

- Adjoint equation

$$
\begin{aligned}
-\partial_t p(x,t) - \text{div}\,(A^T \nabla p) - \text{div}(\mathbf{b}p) &= (u - u_d) \quad \text{in } \Omega \times (0,T) \\
p &= 0, \quad \text{on } \Gamma_D \times (0,T), \\
(A\nabla p + \mathbf{b}p) \cdot \boldsymbol{\nu} &= 0 \quad \text{on } \Gamma_N \times (0,T) \\
p(x,T) &= 0 \quad \text{in } \Omega.
\end{aligned}
$$

- Gradient

$$
(\nabla \mathcal{J}(z), h)_Z = \langle \mathcal{J}'(z), h \rangle_{Z^*,Z} = \langle p + \lambda z, h \rangle_{Z^*,Z} \quad h \in Z.
$$

where $Z = L^2(0,T; L^2(\Omega))$.

Outline

Linear elliptic PDE and Weak Form

Literature: [Braess, 1997, Ciarlet, 2002, Brenner and Scott, 2008, Ern and Guermond, 2004]

Poisson's equation. Let Ω be a bounded open Lipschitz domain with boundary $\partial\Omega$. Consider the following elliptic boundary value problem where given f we seek u solving

$$\begin{aligned} -\mathrm{div}(A\nabla u) &= f \quad \text{in } \Omega \\ u &= g_D \quad \text{on } \partial\Omega_D \\ A\nabla u \cdot \boldsymbol{\nu} &= g_N \quad \text{on } \partial\Omega_N . \end{aligned}$$

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Weak solution. We define

$$V := \{v \in H^1(\Omega) \,:\, v|_{\partial\Omega_D} = g_D\}, \quad V_0 := \{v \in H^1(\Omega) \,:\, v|_{\partial\Omega_D} = 0\}.$$

We seek $u \in V$ such that

$$\int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} g_N v ds, \quad \forall v \in V_0.$$

Weak form cont.

Assume that there exist $w \in H^1(\Omega)$ such that $w|_{\partial\Omega_D} = g_D$. Then we write $u = u_0 + w$ with $u_0|_{\partial\Omega_D} = 0$.

Thus we are reduced to finding $u_0 \in V_0$ solving:

$$\int_\Omega A\nabla u_0 \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} g_N v ds - \int_\Omega A\nabla w \cdot \nabla v dx, \quad \forall v \in V_0.$$

Then $u = u_0 + w$.

- **Weak form.** Find $u_0 \in V_0$ solving:

$$\int_\Omega A\nabla u_0 \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} g_N v ds - \int_\Omega A\nabla w \cdot \nabla v dx, \quad \forall v \in V_0.$$

- **Galerkin approximation.** Find subspace $V_h \subset V_0$ with basis $\{v_1, \ldots, v_n\}$. Approximate u_0 by $u_h = \sum_{i=1}^n u_j v_j$.

  Find $u_h \in V_h$ such that $u_h \in V_h$ solves

$$\int_\Omega A\nabla u_h \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} g_N v ds - \int_\Omega A\nabla w \cdot \nabla v dx, \quad \forall v \in V_h$$

  or equivalently

$$\int_\Omega A\nabla u_h \cdot \nabla v_i dx = \int_\Omega f v_i dx + \int_{\partial\Omega_N} g_N v_i ds - \int_\Omega A\nabla w \cdot \nabla v_i dx, \quad i = \overline{1, n}.$$

Galerkin Approximation

Find $u_h \in V_h$ solving:

$$\int_\Omega A\nabla u_h \cdot \nabla v_i dx = \int_\Omega f v_i dx + \int_{\partial\Omega_N} g_N v_i ds - \int_\Omega A\nabla w \cdot \nabla v_i dx, \quad i = \overline{1,n}.$$

Equivalently we can write a matrix vector system:

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$ with entries

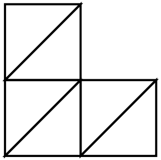
$$\begin{aligned}\mathbf{A}_{ij} &= \int_\Omega A\nabla v_j \cdot \nabla v_i \\ \mathbf{b}_i &= \int_\Omega f v_i dx + \int_{\partial\Omega_N} g_N v_i ds - \int_\Omega A\nabla w \cdot \nabla v_i dx, \quad i = \overline{1,n}.\end{aligned}$$

Piecewise linear basis functions on triangles

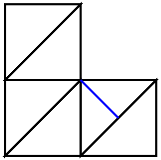
- We haven't specified V_h or basis functions $v_1, \ldots, v_n$ yet.
- We will define V_h as a subspace with piecewise polynomials.
- For simplicity let us consider $\Omega \subset \mathbb{R}^2$, and consider the subspace V_h of functions that are continuous on $\overline{\Omega}$ and piecewise linear basis functions on triangles.

Assume polygonal domain Ω and subdivide into open triangles $K_1, \ldots, K_{\text{elem}}$ such that

$$
\begin{aligned}
\overline{\Omega} &= \cup_i \overline{K}_i, \\
K_i \cap K_j &= \emptyset, \quad i \neq j, \\
\overline{K}_i \cap \overline{K}_j &= \emptyset, \text{ or } = \text{a vertex, or } = \text{ an edge of both } K_i \text{ and } K_j.
\end{aligned}
$$



Admissible triangulation



Inadmissible triangulation

- Subspace V_h consists of functions that are globally continuous on $\overline{\Omega}$ and piecewise linear on the triangles.
- Let $\mathbf{n}_1, \ldots, \mathbf{n}_{n_n} \in \overline{\Omega}$ are the nodes (same as vertices for piecewise linear elements). Here n_n is the number of nodes in the mesh.
- Basis functions $N_1, \ldots, N_{n_n}$ are piecewise linear on triangles and satisfy

$$N_i(\mathbf{n}_j) = \delta_{ij}.$$

  (refer to as *nodal basis*).
- Notice that $N_i(x)$ is nonzero only if x is contained in a element K that contains the node $\mathbf{n}_i$, i.e.,

$$N_i(x) = 0 \quad \text{for } x \notin \cup_{\mathbf{n}_i \in \overline{K}} \overline{K}.$$

- For notation simplicity we assume that $\mathbf{n}_{n+1}, \ldots, \mathbf{n}_{n_n}$ are the nodes corresponding to the Dirichlet boundary conditions, i.e.,
  $\mathbf{n}_1, \ldots \mathbf{n}_n \notin \partial\Omega_D$,
  $\mathbf{n}_{n+1}, \ldots \mathbf{n}_{n_n} \in \partial\Omega_D$.
- The subspace V_h has basis $\{N_1, \ldots, N_n\}$.
- Thus the Galerkin approximation now reads

$$\mathbf{A}\mathbf{u} = \mathbf{b},$$

where

$$\begin{aligned}
\mathbf{A}_{ij} &= \int_\Omega A\nabla N_j \cdot \nabla N_i \\
\mathbf{b}_i &= \int_\Omega f N_i dx + \int_{\partial\Omega_N} g_N N_i ds - \int_\Omega A\nabla w \cdot \nabla N_i dx, \quad i = \overline{1, n}.
\end{aligned}$$

- We will incorporate the Dirichlet boundary conditions later.
- Let us first compute $\mathbf{A} \in \mathbb{R}^{n_n,n_n}$ with entries

$$\mathbf{A}_{ij} = \int_\Omega A\nabla N_j \cdot \nabla N_i$$
$$\mathbf{b}_i = \int_\Omega f N_i dx + \int_{\partial\Omega_N} g_N N_i ds.$$

- We notice that

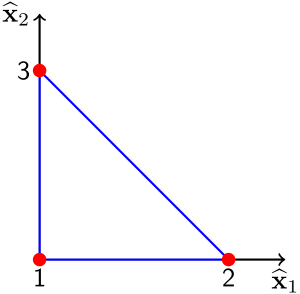
$$\int_\Omega A\nabla N_j \cdot \nabla N_i = \sum_{\ell=1}^{n_{\text{elem}}} \int_{K_\ell} A\nabla N_j \cdot \nabla N_i$$

and

$$\int_{K_\ell} A\nabla N_j \cdot \nabla N_i = 0$$

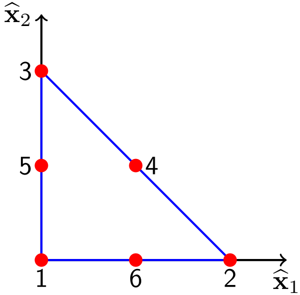
if node $\mathbf{n}_i$ or $\mathbf{n}_j$ is not in $\overline{K}_\ell$.

- For piecewise linear basis functions there are $n_{\text{basis}} = 3$ nodes per element. For each element K_ℓ compute $n_{\text{basis}} \times n_{\text{basis}}$ integrals. The resulting matrices are called *local*.
- Compute these integrals by transforming onto a reference element $\widehat{K}$ with vertices $(0,0),(1,0),(1,1)$.
- The linear basis on the reference triangle are given by



$$
\begin{aligned}
\widehat{N}_1(\widehat{x}) &= 1 - \widehat{x}_1 - \widehat{x}_2 \\
\widehat{N}_2(\widehat{x}) &= \widehat{x}_1 \\
\widehat{N}_3(\widehat{x}) &= \widehat{x}_2
\end{aligned}
$$

Quadratic Basis Functions on Reference Triangle



$$
\begin{aligned}
\widehat{N}_1(\widehat{x}) &= (1-\widehat{x}_1-\widehat{x}_2)(1-2\widehat{x}_1-2\widehat{x}_2)\\
\widehat{N}_2(\widehat{x}) &= \widehat{x}_1(2\widehat{x}_1-1)\\
\widehat{N}_3(\widehat{x}) &= \widehat{x}_2(2\widehat{x}_2-1)\\
\widehat{N}_4(\widehat{x}) &= 4\widehat{x}_1\widehat{x}_2\\
\widehat{N}_5(\widehat{x}) &= 4\widehat{x}_2(1-\widehat{x}_1-\widehat{x}_2)\\
\widehat{N}_6(\widehat{x}) &= 4\widehat{x}_1(1-\widehat{x}_1-\widehat{x}_2)
\end{aligned}
$$

- **Element transformation.** The map

$$
\begin{aligned}
F_E : \widehat{K} &\to K, \\
\widehat{x} &\mapsto F_K(x) = \mathbf{v}_{k_1} + B\widehat{x}
\end{aligned}
$$

  where $B = (\mathbf{v}_{k_2} - \mathbf{v}_{k_1}, \mathbf{v}_{k_3} - \mathbf{v}_{k_1}) \in \mathbb{R}^{2\times 2}$ is bijective and maps vertex $\widehat{\mathbf{v}}_i$ to $\mathbf{v}_{k_i}$, $i = 1, 2, 3$.
- The basis function on element K is given by

$$
N_{k_i}(x) = \widehat{N}_i(F_K^{-1}(x)), \quad i = 1, \ldots, n_{\text{basis}}.
$$

- Derivative. $\nabla N_{k_i}(x) = B^{-T} \nabla \widehat{N}_i(\widehat{x})$.

Assembly

- Let K be an element with nodes $\mathbf{n}_{k_1}, \ldots, \mathbf{n}_{k_{n_{\text{basis}}}}$.
- The integral
$$\int_K A(x) \nabla N_j(x) \cdot \nabla N_i(x)\, dx$$
is only nonzero if $i, j \in \{k_1, \ldots, k_{n_{\text{basis}}}\}$.
- If $i = k_\ell$ and $j = k_m$ then

$$\begin{aligned}\int_K A(x) \nabla N_j(x) \cdot \nabla N_i(x)\, dx &= \int_K A(x) \nabla N_{k_m}(x) \cdot \nabla N_{k_\ell}(x)\, dx \\ &= |\mathsf{det}(B)| \int_{\widehat{K}} A(F_K(\widehat{x})) \Big(B^{-T} \nabla \widehat{N}_m\Big) \Big(B^{-T} \nabla \widehat{N}_\ell\Big)\end{aligned}$$

Dirichlet Boundary Conditions

- Let n_n be the number of nodes in the mesh.
- $\mathbf{n}_{n+1}, \ldots, \mathbf{n}_{n_n}$ be the Dirichlet nodes. The subspace V_h has basis $\{N_1, \ldots, N_n\}$.
- So far we have assembled $\mathbf{A} \in \mathbb{R}^{n_n \times n_n}$ and $\mathbf{b} \in \mathbb{R}^{n_n}$.
- Approximate w by

$$w(x) \approx \sum_{j=n+1}^{n_n} d(\mathbf{n}_j) N_j(x)$$

  (Interpolation at nodes).
- We then modify $\mathbf{b}$ to incorporate nonzero boundary conditions as

$$\mathbf{b}_i = \int_\Omega f N_i + \int_{\partial\Omega} g_N N_i ds - \sum_{j=n+1}^{n_n} \mathbf{A}_{ij} d(\mathbf{n}_j).$$

- Then we need to solve

$$\mathbf{A}(1:n, 1:n)\mathbf{u}(1:n) = \mathbf{b}(1:n),$$

  where $1:n$ corresponds to the free nodes (nodes at which no Dirichlet conditions are given).

Outline

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