

$$(i) (u, u) \geq 0, (u, u) = 0 \Leftrightarrow u = 0$$

$$(ii) (u, v) = (v, u)$$

$$(iii) (u_1 + u_2, v) = (u_1, v) + (u_2, v)$$

$$(iv) (\lambda u, v) = (u, \lambda v) = \lambda (u, v).$$

If (\cdot, \cdot) is a scalar product on H then $\{H, (\cdot, \cdot)\}$ is called a pre-Hilbert space.

Ex ① \mathbb{R}^n is pre-Hilbert with $(u, v) := v^T u$, $u, v \in \mathbb{R}^n$.

② $C_{L^2}[a, b]$ is pre-Hilbert

with $(u, v) := \int_a^b u(t) v(t) dt$.

- Every pre-Hilbert space is a normed linear space w.r.t. its natural norm

$$\|u\| := \sqrt{(u, u)}.$$

Why?

$$(i) \|u\| = \sqrt{(u, u)} \geq 0 \text{ with equality iff } u = 0$$

$$(ii) \|u+v\|^2 = (u+v, u+v)$$

$$= (u, u) + 2(u, v) + (v, v)$$

$$\xrightarrow{\text{By Cauchy-Schwarz}} \leq (u, u) + 2|(u, v)| + (v, v)$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

(iii)

$$\|\lambda u\| = \sqrt{(\lambda u, \lambda u)} = \sqrt{\lambda^2 (u, u)} = |\lambda| \|u\|.$$

* Cauchy-Schwarz

$$|(u, v)| \leq \|u\| \|v\| \text{ with equality iff } u = cv, c \in \mathbb{R}.$$

Def A pre-Hilbert space $\{H, (\cdot, \cdot)\}$ is called a Hilbert space if it is complete w.r.t. the norm

$$\|u\| := \sqrt{(u, u)}, \quad u \in H.$$

Ex ① \mathbb{R}^n with the standard scalar product $(u, v) = v^T u, \|u\| = \sqrt{(u, u)}$

$$\begin{aligned} &= \sqrt{u^T u} \\ &= \left(\sum_{i=1}^n u_i^2 \right)^{1/2} \end{aligned}$$

2-2 Sobolev space: General assumption
 $E \subset \mathbb{R}^n$ a non-empty - bounded - measurable

Def We denote by $L^p(E)$, $1 \leq p < \infty$
 the linear space of all equivalence classes of Lebesgue measurable functions y that satisfy

$$\int_E |y(x)|^p dx < \infty.$$

with the norm

$$\|y\|_{L^p(E)} = \left(\int_E |y(x)|^p dx \right)^{1/p}$$

~~$L^p(E)$~~ with $1 \leq p \leq \infty$ is
 Banach space which is reflexive.
 for $p \in (1, \infty)$.

Def $L^\infty(E)$ denotes the Banach space
 of all equivalence class of Lebesgue
 measurable and essentially bounded
 functions with the norm

$$\|y\|_{L^\infty(E)} = \text{ess sup}_{x \in E} |y(x)| := \inf_{|F|=0} (\sup_{x \in E \setminus F} |y(x)|)$$

Ex $y: [0, 1] \rightarrow \mathbb{R}$ defined by

$$y(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x \in \{1\} \end{cases}$$

$$\|y\|_{L^\infty(E)} = \inf_{|F|=0} (\sup_{x \in E \setminus F} |y(x)|)$$

$$\text{However } \max_{x \in [0, 1]} |y(x)| = 1.$$

General assumption: $\Omega \subset \mathbb{R}^n$ is a domain i.e. open & connected set, $\partial\Omega = \Gamma$, $\bar{\Omega} = \text{cl}(\Omega)$.

Def) Let $k \in \mathbb{N}$. We denote by

$C^k(\Omega)$ the linear space of all real valued functions on Ω that together with their partial derivatives up to k are continuous on Ω .

(iii) The set $\text{supp } v = \overline{\{x \in \Omega : v(x) \neq 0\}}$ is called the support of v .

(iv) $C_0^k(\Omega)$, $k \in \mathbb{N} \cup \{0\}$, denotes the set of all k -times continuously differentiable functions with compact support in Ω .

- $C_0^\infty(\Omega)$ is called the space of test functions.

- Notation: Multi-indices is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \geq 0$ integers. The number $|\alpha| = \alpha_1 + \dots + \alpha_n$ is called the length of the multi index.

The component of α i.e α_i denotes the $x_i^{\alpha_i}$ derivative of the function $f: \Omega \rightarrow \mathbb{R}$ w.r.t x_i .

Ex $\Omega \subset \mathbb{R}^n$, $f: \Omega \rightarrow \mathbb{R}$ defined by

$$f(x, y, z) = x^y z^2, \quad \alpha = (1, 2, 1)$$

$$D^\alpha f = \frac{\partial^4 f}{\partial x \partial y^2 \partial z} = 2 \cdot \cancel{8}$$



Def Let $\Omega \subset \mathbb{R}^n$ be bounded. For any $k \in \mathbb{N} \cup \{0\}$, we denote by $C^k(\bar{\Omega})$ the linear space of all elements of $C^k(\Omega)$ that together with their partial derivatives up to k can be continuously extended to $\bar{\Omega}$. In the case $k=0$, we write $C(\bar{\Omega})$ instead of $C^0(\bar{\Omega})$.

The space $C^k(\bar{\Omega})$ are Banach spaces with respect to the following norms:-

$$\|y\|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |y(x)|$$

$$\|y\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \leq k} \|D^\alpha y\|_{C(\bar{\Omega})}$$

* Regular domains ("Lipschitz domains")

Def: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega = \Gamma$. We say Ω or Γ belong to the class $C^{k,1}$, $k \in \mathbb{N} \cup \{0\}$ if there exists finitely many local coordinate systems s_1, \dots, s_N , functions h_1, \dots, h_M and numbers $a > 0$, $b > 0$ that have the following properties: