

## 2.1 Normed space

## 2.2 Sobolev space

Def: Let  $X$  be a linear space over  $\mathbb{R}$ , i.e.  $\forall x_1, x_2, x_3 \in X$  the following satisfied  
 Associativity - Commutativity - identity  
 element - inverse element for "+" - compatibility  
 of scalar - distributivity of scalar

A mapping  $\|\cdot\|: X \rightarrow \mathbb{R}$  is called a norm on  $X$  if  $\forall x, y \in X, \lambda \in \mathbb{R}$ :

$$(i) \quad \|x\| > 0 \quad \forall x \neq 0, \|x\| = 0 \Leftrightarrow x = 0$$

$$(ii) \quad \|x+y\| \leq \|x\| + \|y\|$$

$$(iii) \quad \|\lambda x\| = |\lambda| \|x\| \quad \text{"homogeneity".}$$

If  $\|\cdot\|$  is a norm in  $X$ , then  $\{x, \|\cdot\|\}$   
 is called a "real" normed space.

Ex ①  $(\mathbb{R}^n, |x|)$  is a normed space  
 where  $|x| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}$  "By Minkowski"  
 $\|\cdot\| := |\cdot|$  for  $\underline{i}$

②  $C[a, b] := \{x: [a, b] \rightarrow \mathbb{R}; x \text{ is continuous}\}$   
 is a normed space with

$$\|x\|_{[a, b]} = \max_{t \in [a, b]} |x(t)|$$

③  $C_{L^2}[a, b]$  i.e.  $C[a, b]$  with  $L^2$ -norm  
 $\|x\|_{C_{L^2}[a, b]} = \left( \int_a^b |x(t)|^2 dt \right)^{1/2}$  "By Minkowski"  
 $\text{for } ii$

Def let  $\{X, \|\cdot\|\}$  be a normed space  
and  $(x_n)_{n \geq 1} \subset X$  be a sequence.  
Then:-

(1)  $(x_n)_{n \geq 1}$  is said to be convergent  
if  $\exists x \in X$  such that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0$$

(2) We call  $x$  the limit of  $(x_n)_{n \geq 1}$

written as  $\lim_{n \rightarrow \infty} x_n = x$ .

(3) The sequence  $(x_n)_{n \geq 1}$  is called Cauchy sequence if  $\forall \varepsilon > 0$

$$\exists n_0 = n_0(\varepsilon) \in \mathbb{N} \text{ s.t } \|x_n - x_m\| < \varepsilon$$

$$\forall n, m > n_0(\varepsilon).$$

Ex Let  $X = C_{L^2}[a, b]$  with  $\|x\|_{L^2[a, b]} = \left( \int_a^b |x(t)|^2 dt \right)^{\frac{1}{2}}$ ,  
 $a=0, b=2$ .

Consider  $(x_n)_{n \geq 1} \subset X$  defined by

$$x_n(t) = \min\{1, t^n\}, t \in [0, 2], n \in \mathbb{N}.$$

$$\begin{aligned} \|x_n - x_m\|^2 &= \int_0^2 |x_n(t) - x_m(t)|^2 dt = \int_0^1 |x_n(t) - x_m(t)|^2 dt \\ &\quad + \int_1^2 |x_n(t) - x_m(t)|^2 dt = \int_0^1 (t^n - t^m)^2 dt + \int_1^2 (1-1)^2 dt \\ &= \int_0^1 (t^{2n} + t^{2m} - 2t^{n+m}) dt = \frac{t^{2n+1}}{2n+1} \Big|_0^1 + \frac{t^{2m+1}}{2m+1} \Big|_0^1 - \frac{2t^{n+m+1}}{n+m+1} \Big|_0^1 \end{aligned}$$

$$= \frac{1}{2n+1} + \frac{1}{2m+1} - \frac{2}{n+m+1} \leq \frac{1}{2m+1} + \frac{1}{2m+1} - \frac{2}{n+m+1}$$

$$\leq \frac{2}{2m+1}$$

for  $n \geq m$ .

Hence, for any  $\epsilon > 0$ , if we choose

$n_0 \in \mathbb{N}$  s.t

$$\frac{2}{2n_0+1} < \epsilon^2, \text{ then for } n, m > n_0 \text{ with } n \geq m$$

$$\|x_n - x_m\|^2 \leq \frac{2}{2m+1} < \frac{2}{2n_0+1} < \epsilon^2$$

$\Rightarrow \|x_n - x_m\|^2 < \epsilon$ . So  $(x_n)_{n \geq 1}$  is Cauchy.

However,

$$\lim_{n \rightarrow \infty} x_n(t) = \begin{cases} 0 & t \in [0, 1] \\ 1 & t \in [1, 2] \end{cases} \notin C_2^{[0,1]}.$$

Def A normed space, is said to be  
 $\Leftrightarrow$  complete if every Cauchy sequence in

complete if every Cauchy sequence in  
 $X$  converges i.e has a limit in  $X$ .

A complete normed space is called  
 Banach space.

Ex ①  $\mathbb{R}^n$  ②  $C_{[a,b]}^{[a,b]}$  with  $\|x\| = \max_{t \in [a,b]} |x(t)|$

Def Let  $H$  be a real linear space.

A mapping  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$  is called a scalar product on  $H$  if the following conditions are satisfied  $\forall u_1, v_1, u_2, v_2 \in H, \lambda \in \mathbb{R}$ :

