Center for Mathematics and Artificial Intelligence

Introduction to Optimal Control Problems with Partial Differential Equations (PDEs)

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Example 1: Rocket car

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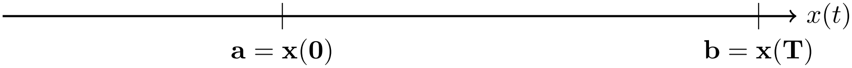

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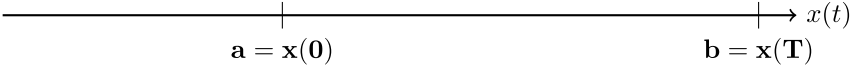

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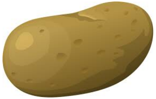
"Minimize T **under the constraint that (SE) is fulfilled and** $|z(t)| \leq 1$**"**

Example 2: Potato



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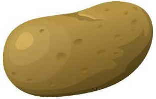

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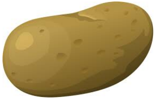

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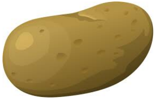

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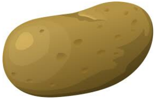

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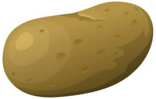

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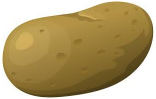

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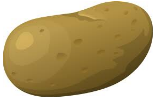

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- assumption: isotropic potato, heat conducting surface
- postulate: model describes the "real world"

Example 2: Potato

state equation

(SE)

$$
\begin{aligned}
u_t(x,t) - \Delta_x u(x,t) &= 0 && \text{in } Q := (0,T] \times \mathit{\Omega} && (1)\\
\partial_n u(x,t) &= \alpha(z(x,t) - u(x,t)) && \text{on } \mathit{\Sigma} := (0,T] \times \mathit{\Gamma} && (2)\\
u(x,0) &= u_0(x) && \text{in } \mathit{\Omega} && (3)
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$\rightarrow$ control the system with respect to a **control goal**

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problem formulation

What is our control goal?
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such that (1),(2),(3) are satisfied

Optimal control problem

control

$\downarrow\downarrow\downarrow$

PROCESS $\Rightarrow$ **desired effect**

Optimal control problem

control
$\downarrow\downarrow\downarrow$
PROCESS $\Rightarrow$ desired effect

Mathematical formulation:

(P) $$\min_{(u,z)\in\mathcal{U}\times\mathcal{Z}} J(u,z) \quad \text{s.t.} \quad e(u,z)=0 \quad \text{and} \quad z\in\mathcal{Z}_{\text{ad}}$$

u: state, z: control, $J:\mathcal{U}\times\mathcal{Z}\to\mathbb{R}$ cost functional
$e:\mathcal{U}\times\mathcal{Z}\to\mathcal{Y}$ operator, $\mathcal{Z}_{\text{ad}}$: admissible control set

Mathematical formulation: Example 2: Potato

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Theoretical analysis

- solvability of the state equation **(SE)**
  - weak formulation
  - control-state-operator $\mathcal{S} : z \mapsto u(z)$

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- solvability of the state equation **(SE)**
  - weak formulation
  - control-state-operator $\mathcal{S} : z \mapsto u(z)$
- solvability of the optimal control problem $(\mathbf{P})$
  - existence
  - uniqueness

First order necessary optimality conditions

$$\text{(P)} \qquad \boxed{\min_{(u,z)} J(u,z) \quad \text{s.t.} \quad e(u,z) = 0}$$

- Let $\mathcal{Z}_{\text{ad}} = L^2(\Omega)$ for simplicity

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$$\mathcal{L}(u,z;p) = J(u,z) + \langle e(u,z), p \rangle_{\mathcal{Y},\mathcal{Y}'}$$

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- Karush-Kuhn-Tucker (KKT) conditions:

(KKT 1) $\mathcal{L}_u(\bar{u},\bar{z};\bar{p})u = 0$ for all u with $u(0) = 0$

(KKT 2) $\mathcal{L}_z(\bar{u},\bar{z};\bar{p})z = 0$ for all $z \in L^2(\Omega)$

(KKT 3) $\mathcal{L}_p(\bar{u},\bar{z};\bar{p})p = 0$ $\rightarrow$ state equation $e(u,z) = 0$!

First order necessary optimality conditions

- **(KKT 1)** $\mathcal{L}_u(\bar{u}, \bar{z}; \bar{p})u = 0$ for all u with $u(0) = 0$

"adjoint equation (AE)"
$$\begin{cases} -\bar{p}_t - \Delta\bar{p} = 0 & \text{in } Q \\ \partial_n \bar{p} = -\alpha\bar{p} & \text{on } \Sigma \\ \bar{p}(T) = \bar{u}(T) - u_d & \text{in } \Omega \end{cases}$$

- **(KKT 2)** $\mathcal{L}_z(\bar{u}, \bar{z}; \bar{p})z = 0$ for all $z \in L^2(\Omega)$

equation with z and p $\gamma\bar{z} - \bar{p} = 0$

- **(KKT 3)** $\mathcal{L}_p(\bar{u}, \bar{z}; \bar{p})p = 0$

"state equation (SE)"
$$\begin{cases} \bar{u}_t - \Delta\bar{u} = 0 & \text{in } Q \\ \partial_n \bar{u} = \alpha(\bar{z} - \bar{u}) & \text{on } \Sigma \\ \bar{u}(\cdot, 0) = u_0 & \text{in } \Omega \end{cases}$$

Second order sufficient optimality conditions

- optimal control problem is a convex problem (i.e. cost functional is convex and constraints form a convex set)

- first order necessary optimality conditions are sufficient ✓

- for non-convex problems (e.g. nonlinear **(SE)**) it is more difficult:
  - uniqueness of a solution?
  - second order sufficient conditions $\rightarrow$ second order derivatives ...?
  - numerical algorithm (Newton, SQP,...)?

Numerical realization

Reduced space approach

- Rewrite $\boxed{\min J(u,z) \text{ s.t. } e(u,z)=0}$ as $\boxed{\min_{z\in\mathcal{Z}} \hat{J}(z)}$
- It can be shown that $\nabla \hat{J} = \gamma z - p.$ $\leftarrow$ **(KKT 2)**

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- **Gradient method:**
  - Choose an initial control z^0
  - **while** (termination criterium is not fulfilled) **do**
    - Compute corresponding state $u \leftarrow$ **(SE)**
    - Compute corresponding adjoint state $p \leftarrow$ **(AE)**
    - Determine stepsize t^k
    - Compute update $z^{k+1} = z^k - t^k \cdot \nabla\hat{J}(z^k)$
  - **end(while)**

Numerical realization

Full space approach

- Full system:

$$F(\bar{u}, \bar{z}; \bar{p}) = \begin{pmatrix} \mathcal{L}_u(\bar{u}, \bar{z}; \bar{p})u \\ \mathcal{L}_z(\bar{u}, \bar{z}; \bar{p})z \\ e(\bar{u}, \bar{z}) \end{pmatrix}$$

- Find $(\bar{u}, \bar{z}; \bar{p})$ such that $F(\bar{u}, \bar{z}; \bar{p}) = 0$
- **Newton's method:**
  - Choose an initial triple (u^0, z^0, p^0)
  - **while** (termination criterium is not fulfilled) **do**
    - Obtain s^k by solving

$$\partial F(u^k, z^k; p^k)s^k = -F(u^k, z^k; p^k)$$

    - Determine stepsize t^k
    - Compute update $z^{k+1} = z^k + t^k s^k$
  - **end(while)**

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Challenges

We have to solve **(SE)** and **(AE)** repeatedly
$\rightarrow$ (time) costly, storage needed

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Remedy:
Replace **(SE)** and **(AE)** with

- model order reduction techniques (i.e. Proper Orthogonal Decomposition)
- Sketching

Literature

- Optimal Control of Partial Differential Equations: Theory, Methods and Applications, *F. Tröltzsch*, AMS 2010
- Optimization with PDE Constraints, *M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich*, Springer 2008